Spherical Astronomy for Worldbuilders

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Chapter 0

Introduction

Astrology, which is a method of divination based on the location of the stars and planets in the sky, can be a very important part of worldbuilding as pretty much every culture has some reference to astrology. The basis of astrology is **Spherical Astronomy**, the science of observing the sky. This post will serve as a guide to spherical astronomy for the worldbuilder, and will start from the basics and build up to calculating the location of objects in the sky, and also calculate various astronomical phenomena such that the worldbuilder can use these calculations in their astrological systems.

NOTE! Some of the derivations of formulae in this guide assumes the reader is familiar with algebra, trigonometry, basic differentiation and integration, basic vector operations, and basic matrix operations. (See §1.1 Useful Mathematics)

0.1 Useful Mathematics

0.1.1 Algebra

$$(a+b)(c+d) = ac + ad + bc + bd \tag{0.1}$$

$$\frac{a}{b} = \frac{c}{d} \implies ad = bc \tag{0.2}$$

$$x^0 = 1 \tag{0.3}$$

$$x^{a/b} = \sqrt[b]{x^a} \tag{0.4}$$

$$x^{-a} = \frac{1}{x^a} \tag{0.5}$$

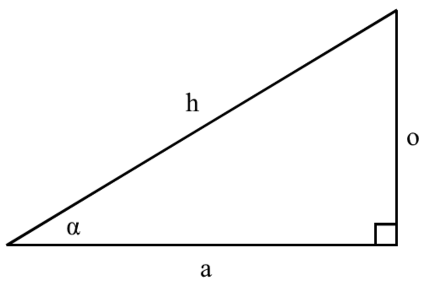
$$x^a \cdot x^b = x^{a+b} \tag{0.6}$$

$$(x^a)^b = x^{ab} \tag{0.7}$$

$$ax^2 + bx + c = 0 \implies x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{0.8}$$

$$\tag{0.9}$$

0.1.2 Trigonometry



$$\sin(\alpha) = \frac{o}{h} \tag{0.10}$$

$$\cos(\alpha) = \frac{a}{h} \tag{0.11}$$

$$\tan(\alpha) = \frac{o}{a} \tag{0.12}$$

$$\mathrm{trig}(\alpha + n \cdot 360^\circ) = \mathrm{trig}(\alpha) \;\; \{n \in \mathbb{Z}\} \tag{0.13}$$

$$\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)} \tag{0.14}$$

$$\csc(\alpha) = \frac{1}{\sin(\alpha)} \tag{0.15}$$

$$\sec(\alpha) = \frac{1}{\cos(\alpha)} \tag{0.16}$$

$$\cot(\alpha) = \frac{1}{\tan(\alpha)} = \frac{\cos(\alpha)}{\sin(\alpha)} \tag{0.17}$$

$$\begin{aligned}\sin(\alpha) &= -\sin(-\alpha) = -\sin(\alpha + 180^\circ) = \sin(180^\circ - \alpha) \\ &= \cos(90^\circ - \alpha) = -\cos(90^\circ + \alpha) = \cos(\alpha - 90^\circ)\end{aligned} \tag{0.18}$$

$$\begin{aligned}\cos(\alpha) &= \cos(-\alpha) = -\cos(\alpha + 180^\circ) = -\cos(180^\circ - \alpha) \\ &= \sin(90^\circ - \alpha) = \sin(90^\circ + \alpha) = -\sin(\alpha - 90^\circ)\end{aligned} \tag{0.19}$$

$$\begin{aligned}\tan(\alpha) &= -\tan(-\alpha) = \tan(\alpha + 180^\circ) = -\tan(180^\circ - \alpha) \\ &= \cot(90^\circ - \alpha) = -\cot(90^\circ + \alpha) = -\cot(\alpha - 90^\circ)\end{aligned} \tag{0.20}$$

$$\sin^2(\alpha) + \cos^2(\alpha) = 1 \tag{0.21}$$

$$\tan^2(\alpha) + 1 = \sec^2(\alpha) \tag{0.22}$$

$$\cot^2(\alpha) + 1 = \csc^2(\alpha) \tag{0.23}$$

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \tag{0.24}$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \tag{0.25}$$

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) \tag{0.26}$$

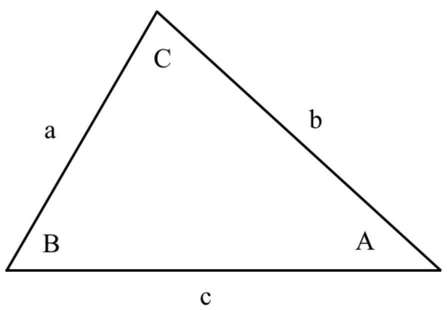
$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) = 2\cos^2(\alpha) - 1 = 1 - 2\sin^2(\alpha) \tag{0.27}$$

$$\sin^2(\alpha) = \frac{1}{2}(1 - \cos(2\alpha)) \tag{0.28}$$

$$\cos^2(\alpha) = \frac{1}{2}(1 + \cos(2\alpha)) \tag{0.29}$$

$$\arctan(y, x) = \begin{cases} \arctan(y/x) & \text{if } x > 0 \\ \arctan(y/x) + 180^\circ & \text{if } x < 0 \text{ and } y \geq 0 \\ \arctan(y/x) - 180^\circ & \text{if } x < 0 \text{ and } y < 0 \\ 90^\circ & \text{if } x = 0 \text{ and } y > 0 \\ -90^\circ & \text{if } x = 0 \text{ and } y < 0 \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0 \end{cases} \tag{0.30}$$

0.1.3 Geometry

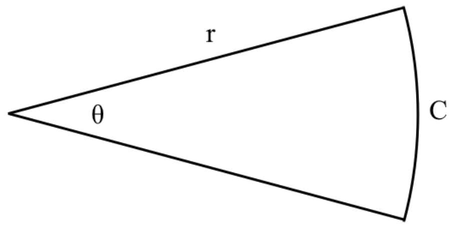


$$A + B + C = 180^\circ \tag{0.31}$$

$$a^2 = b^2 + c^2 - 2bc\cos(A) \tag{0.32}$$

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)} \tag{0.33}$$

$$S = \frac{1}{2}ab\sin(C) \tag{0.34}$$



(θ in radians)

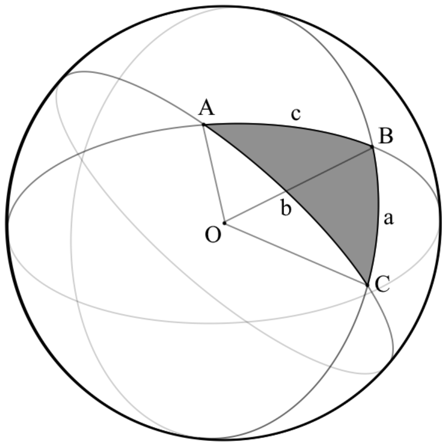
$$C = r\theta \tag{0.35}$$

$$S = \frac{1}{2}r^2\theta \tag{0.36}$$

$$1 \text{ rev} = 2\pi \text{ rad} = 360^\circ = 24^h \tag{0.37}$$

$$1^\circ = 60' = 3600'' \tag{0.38}$$

$$1^h = 60^m = 3600^s \tag{0.39}$$



A, B, C are points on a sphere with center O. Let the angles BAC, ABC, and BCA be denoted A, B, and C respectively, and the angles BOC, AOC, and AOB be denoted a, b, and c respectively.

$$A + B + C \geq 180^\circ \tag{0.40}$$

$$\cos(a) = \cos(b)\cos(c) + \sin(b)\sin(c)\cos(A) \tag{0.41}$$

$$\cos(A) = -\cos(B)\cos(C) + \sin(B)\sin(C)\cos(a) \tag{0.42}$$

$$\sin(a)\cos(B) = \cos(b)\sin(c) - \sin(b)\cos(c)\cos(A) \tag{0.43}$$

$$\sin(A)\cos(b) = \cos(B)\sin(C) - \sin(B)\cos(C)\cos(a) \tag{0.44}$$

$$\frac{\sin(a)}{\sin(A)} = \frac{\sin(b)}{\sin(B)} = \frac{\sin(c)}{\sin(C)} \tag{0.45}$$

$$\cos(a)\cos(C) = \sin(a)\cot(b) - \sin(C)\cot(B) \tag{0.46}$$

$$\cos(A)\cos(c) = \sin(A)\cot(B) - \sin(c)\cot(b) \tag{0.47}$$

0.1.4 Calculus

$$\begin{aligned} \frac{d}{dx}f(x) &= f'(x) = \dot{f} \\ &= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} \end{aligned} \tag{0.48}$$

$$\frac{d}{dx}a = 0 \tag{0.49}$$

$$\frac{d}{dx}x^a = ax^{a-1} \tag{0.50}$$

$$(x \text{ in radians})$$

$$\frac{d}{dx}\sin(x) = \cos(x) \tag{0.51}$$

$$\frac{d}{dx}\cos(x) = -\sin(x) \tag{0.52}$$

$$\frac{d}{dx}\tan(x) = \sec^2(x) \tag{0.53}$$

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}} \tag{0.54}$$

$$\frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1-x^2}} \tag{0.55}$$

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2} \tag{0.56}$$

$$\frac{d}{dx}[af(x) + bg(x)] = af'(x) + bg'(x) \tag{0.57}$$

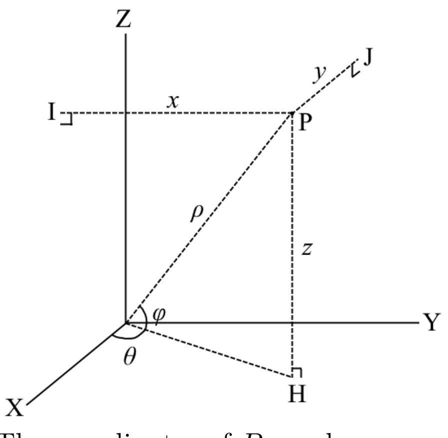
$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x) \tag{0.58}$$

$$\frac{dy}{dx} = \frac{dy}{du_1}\frac{du_1}{du_2}\frac{du_2}{du_3}\cdots\frac{du_{n-1}}{du_n}\frac{du_n}{dx} \tag{0.59}$$

$$\therefore \frac{d}{dx}[f(g(x))] = f'(g(x))g'(x) \tag{0.60}$$

$$\int_a^b f'(x)dx = f(b) - f(a) \tag{0.61}$$

0.1.5 3D Coordinates



O is the origin. Lines OX, OY, and OZ are the x, y, and z axes respectively and are all perpendicular to each other. Planes OXY, OXZ, and OYZ are the xy, xz, and yz planes respectively and are all perpendicular to each other. H, I, J are the orthogonal projection of P onto the xy, xz, and yz planes respectively.

The coordinates of P can be expressed in **cartesian** (or **rectangular**) coordinates as:

$$x = IP \quad y = JP \quad z = HP$$

or in **spherical** coordinates as:

$$\rho = OP \quad \theta = XOH \quad \phi = HOP$$

The position vector $\mathbf{r}$ is given as:

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{0.62}$$

0.1.6 Vectors

$\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ are the unit (right-handed) cartesian basis vectors of $\mathbb{R}^3$.
θ is the angle between $\mathbf{v}$ and $\mathbf{u}$.
$\mathbf{0}$ is the zero vector.

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_1\mathbf{e}_x + v_2\mathbf{e}_y + v_3\mathbf{e}_z \tag{0.63}$$

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^3} \tag{0.64}$$

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|} \tag{0.65}$$

$$a\mathbf{v} + b\mathbf{u} = \begin{bmatrix} av_1 + bu_1 \\ av_1 + bu_2 \\ av_3 + bu_3 \end{bmatrix} \tag{0.66}$$

$$\mathbf{v} \cdot \mathbf{u} = v_1u_1 + v_2u_2 + v_3u_3 \tag{0.67}$$

$$= \mathbf{u} \cdot \mathbf{v} \tag{0.68}$$

$$= |\mathbf{v}||\mathbf{u}| \cos(\theta) \tag{0.69}$$

$$\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2 \tag{0.70}$$

$$\mathbf{v} \times \mathbf{u} = \begin{bmatrix} v_2u_3 - v_3u_2 \\ v_3u_1 - v_1u_3 \\ v_1u_2 - v_2u_1 \end{bmatrix} \tag{0.71}$$

$$= -(\mathbf{u} \times \mathbf{v}) \tag{0.72}$$

$$|\mathbf{v} \times \mathbf{u}| = |\mathbf{v}||\mathbf{u}| \sin(\theta) \tag{0.73}$$

$$\mathbf{v} \times \mathbf{v} = \mathbf{0} \tag{0.74}$$

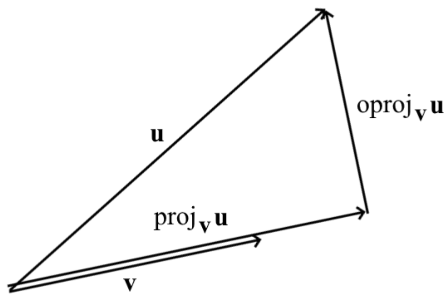
$$\mathbf{v} \cdot (\mathbf{v} \times \mathbf{u}) = 0 = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{u}) \tag{0.75}$$

$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{w}) = \mathbf{u} \cdot (\mathbf{w} \times \mathbf{v}) = \mathbf{w} \cdot (\mathbf{v} \times \mathbf{u}) \tag{0.76}$$

$$\mathbf{v} \times (\mathbf{u} \times \mathbf{w}) = (\mathbf{v} \cdot \mathbf{w})\mathbf{u} - (\mathbf{v} \cdot \mathbf{u})\mathbf{w} \tag{0.77}$$

$$(\mathbf{v} \times \mathbf{u}) \times \mathbf{w} = (\mathbf{v} \cdot \mathbf{w})\mathbf{u} - (\mathbf{u} \cdot \mathbf{w})\mathbf{v} \tag{0.78}$$

Orthogonal Projections of Vectors onto other Vectors:



$$\mathrm{proj}_{\mathbf{v}}\mathbf{u} = (\mathbf{v} \cdot \mathbf{u})\hat{\mathbf{v}} = \frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v} = \frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v} \cdot \mathbf{v}}\mathbf{u} \tag{0.79}$$

$$\mathrm{oproj}_{\mathbf{v}}\mathbf{u} = \mathbf{u} - \mathrm{proj}_{\mathbf{v}}\mathbf{u} \tag{0.80}$$

$$\mathbf{v} \cdot \mathrm{proj}_{\mathbf{v}}\mathbf{u} = \mathbf{v} \cdot \mathbf{u} \tag{0.81}$$

$$\mathbf{v} \cdot \mathrm{oproj}_{\mathbf{v}}\mathbf{u} = 0 \tag{0.82}$$

(0.83)

0.1.7 Matrices

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \tag{0.84}$$

$$cA + dB = \begin{bmatrix} ca_1 + db_1 & ca_2 + db_2 & ca_3 + db_3 \\ ca_4 + db_4 & ca_5 + db_5 & ca_6 + db_6 \\ ca_7 + db_7 & ca_8 + db_8 & ca_9 + db_9 \end{bmatrix} \tag{0.85}$$

$$A \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} a_1 v_1 + a_2 v_2 + a_3 v_3 \\ a_4 v_1 + a_5 v_2 + a_6 v_3 \\ a_7 v_1 + a_8 v_2 + a_9 v_3 \end{bmatrix} \tag{0.86}$$

$$AB = \begin{bmatrix} a_1 b_1 + a_2 b_4 + a_3 b_7 & a_1 b_2 + a_2 b_5 + a_3 b_8 & a_1 b_3 + a_2 b_6 + a_3 b_9 \\ a_4 b_1 + a_5 b_4 + a_6 b_7 & a_4 b_2 + a_5 b_5 + a_6 b_8 & a_4 b_3 + a_5 b_6 + a_6 b_9 \\ a_7 b_1 + a_8 b_4 + a_9 b_7 & a_7 b_2 + a_8 b_5 + a_9 b_8 & a_7 b_3 + a_8 b_6 + a_9 b_9 \end{bmatrix} \tag{0.87}$$

$$A^T = \begin{bmatrix} a_1 & a_4 & a_7 \\ a_2 & a_5 & a_8 \\ a_3 & a_6 & a_9 \end{bmatrix} \tag{0.88}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}^T = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \tag{0.89}$$

$$AB \neq BA \tag{0.90}$$

$$\left(A^T\right)^T = A \tag{0.91}$$

$$(AB)^T = B^T A^T \tag{0.92}$$

Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors in $\mathbb{R}^3$.

$$\mathbf{a}^T \mathbf{b} = \mathbf{a} \cdot \mathbf{b} \tag{0.93}$$

$$\mathbf{a}\mathbf{b}^T = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} \tag{0.94}$$

Coordinate Transformation Rotation Matrices:
Let $B_{\text{old}} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ and $B_{\text{new}} = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ be two bases of $\mathbb{R}^3$. Say $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are expressed in terms of B_{old} as:

$$\mathbf{b}_1 = c_{11}\mathbf{a}_1 + c_{21}\mathbf{a}_2 + c_{31}\mathbf{a}_3$$
$$\mathbf{b}_2 = c_{12}\mathbf{a}_1 + c_{22}\mathbf{a}_2 + c_{32}\mathbf{a}_3$$
$$\mathbf{b}_3 = c_{13}\mathbf{a}_1 + c_{23}\mathbf{a}_2 + c_{33}\mathbf{a}_3$$

Which can also be written as:

$$\mathbf{b}_j = \sum_{i=1}^{3} c_{ij}\mathbf{a}_i$$

Now if a vector $\mathbf{v}$ in $\mathbb{R}^3$ has coordinates (x_1, x_2, x_3) in terms of B_{old} and coordinates (y_1, y_2, y_3) in terms of B_{new}, we can say that:

$$\mathbf{v} = \sum_{i=1}^{3} x_i \mathbf{a}_i = \sum_{j=1}^{3} y_j \mathbf{b}_j$$

Thus

$$\sum_{i=1}^{3} x_i \mathbf{a}_i = \sum_{j=1}^{3} y_j \sum_{i=1}^{3} c_{ij} \mathbf{a}_i$$
$$= \sum_{i=1}^{3} \sum_{j=1}^{3} c_{ij} y_j \mathbf{a}_i$$
$$\therefore x_i = \sum_{j=1}^{3} c_{ij} y_j$$

If we put x_1, x_2, x_3 into a column vector $\mathbf{x}$ and y_1, y_2, y_3 into a column vector $\mathbf{y}$, we can write

$$\mathbf{x} = M\mathbf{y}$$

where M is given by:

$$M = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

Thus we can say:

$$\mathbf{y} = M^{-1}\mathbf{x}$$

If M is a rotation matrix, M^{-1} is simply M^T because rotation matrices are orthogonal.

Thus if we want to rotate our coordinate frame by an angle θ about the x, y, or z axes, we can say:

$$\mathbf{v}_{\text{New Coordinate Frame}} = R_n R_{n-1} \cdots R_2 R_1 \mathbf{v}_{\text{Old Coordinate Frame}} \tag{0.95}$$

And

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix} \tag{0.96}$$

$$R_y = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \tag{0.97}$$

$$R_z = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{0.98}$$

Where R_x denotes a rotation by θ about the x axis, R_y denotes one about the y axis, and R_z denotes one about the z axis.

Chapter 1

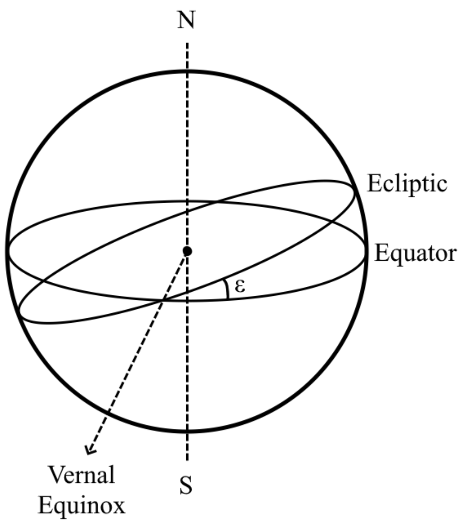
Coordinates

An ephemeris (plural *ephemerides*) is a table listing the locations of celestial objects at specific times. The process of calculating the ephemeris will be shown over the course of chapters 1 to 3.

1.1 The Celestial Sphere

Let us begin by talking about how objects are even located in space. The most important thing in spherical astronomy is the location of celestial objects as seen from Earth (from a geocentric perspective). From the Earth, the sky appears to be a great dome all around us, and this is called the Celestial Sphere. This is a large sphere of arbitrary radius (commonly just put to 1) that surrounds the Earth that all the stars and planets are projected onto. There are two lines (circles in actuality) of great importance on the Celestial Sphere.

- **The Celestial Equator**
  - This is the circle on the celestial sphere obtained by projecting the Earth's equator onto the celestial sphere. If one were standing on the equator of the Earth, the celestial equator would appear to be a great circle passing right above the observer, going from East to West.
- **The Ecliptic**
  - This is the circle the Sun appears to make in the sky over the course of a year. In other words, it is the *plane of the orbit of the Earth*. Due to the axial tilt (ε) of the Earth, the ecliptic and the celestial equator are tilted from each other by ε. Because the Solar System is more or less flat, all the planets, including the Moon, more or less lie on this ecliptic line.



Because the ecliptic is tilted with respect to the equator, there are two points at which these two great circles meet. The point at which the ecliptic goes from being below the equator to above the equator is known as the Cusp of Aries (also known as the Vernal Equinox or the First Point of Aries, this will be referred to simply as "Aries" in this guide), which, ironically, now lies in Pisces due to the slow (a period of about 26000 years) precession of the axial tilt of the Earth. (*For the purposes of worldbuilding, this "axial precession" will be ignored.*) This point is of special importance as it is the place at which almost all angular measurements are made with respect to.

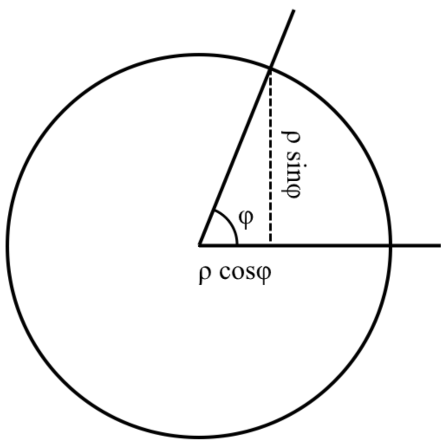
This point is called the *Equinox point* because, if the Sun is located at it, the sun is passing directly over the equator, and the length of daytime will be exacly 1/2 of a day (12 hours) all across the globe. The *Vernal* comes from the fact that, because the Sun is travelling towards the Northern direction, it is Spring time in the Northern Hemisphere when this event occurs (Vernal means Spring in Latin).

1.2 Coordinate Axes

Evidently the best way to locate a point on the Celestial *Sphere* is with *spherical* coordinates. This is a coordinate system based on three values:

$$\begin{aligned} \varphi &= \text{the vertical angle,} \\ \theta &= \text{the horizontal angle,} \\ \rho &= \text{the radius.} \end{aligned}$$

ρ is the distance from the origin to the point in question, θ is the angle East from North, and φ is the angle of altitude from the xy-plane. (Note in mathematics and physics (and the diagram on Wikipedia), instead of φ being the angle above the xy-plane, it is the complement of that angle, the angle down from the z-axis.) Thus, θ ranges from 0° to 360°, and φ from −90° to 90°.
This coordinate system can be thought of as the longitude-latitude system of Earth, where θ is the longitude, and φ is the latitude. Using trigonometry, one can find the radius of the latitude circle of a sphere with radius ρ is $\rho\cos(\varphi)$ (view figure), so:



$$\left.\begin{aligned} x &= \rho\cos(\varphi)\cos(\theta) \\ y &= \rho\cos(\varphi)\sin(\theta) \\ z &= \rho\sin(\varphi) \end{aligned}\right\} \quad (1.1)$$

Where x, y, and z are the cartesian coordinates of the point described by ρ, θ, and φ.
The reverse transformation, also derived by simple geometry, is given below:

$$\left.\begin{aligned} \rho &= \sqrt{x^2+y^2+z^2} \\ \theta &= \arctan(y,x) \\ \varphi &= \arcsin(z/\rho) \end{aligned}\right\} \quad (1.2)$$

Where $\arctan(y,x)$ is the two argument arctangent, used to avoid tangent ambiguity. Note that while most calculators and programming languages use $\arctan(y,x)$, *WolframAlpha* uses $\arctan(x,y)$.
There are two coordinate systems in wide use. The **Equatorial Coordinate System** and the **Ecliptic Coordinate System**.

- **The Equatorial Coordinate System**
  - In the Equatorial Coordinate System, the angles θ and φ are called the *Right Ascension* and *Declination* and are denoted α and δ respectively.
  - The Right Ascension is measured in the plane of the Celestial Equator from the Cusp of Aries with East as the positive direction. Declination is measured perpendicular from the Celestial Equator such that positive is North.

- The Right Ascension has a peculiar unit, the angle is measured by *hours*, *minutes*, and *seconds*, where 24 hours is one revolution, 60 minutes is one hour, and 60 seconds is one minute. (Thus, 1 hour is $15°$, and 1 minute is $0.25°$, and 1 second is $0.004166...°$.)

- The points of $+90°$ and $-90°$ Declination are called the *North* and *South Celestial Poles* respectively. These points can be thought of as the points in the sky right above True North and True South of Earth. From the view of an observer on the Earth, the celestial poles are always in the direction of North and South. Indeed, the reason the North Star always points North is because it is located so close to the North Celestial Pole (Declension $+89°16'$).

- The Earth rotates around the axis that connects the two celestial poles.

• **The Ecliptic Coordinate System**

- In the Ecliptic Coordinate System, the angles θ and φ are called the *Ecliptic Longitude* and *Ecliptic Latitude* and are denoted λ and β respectively.

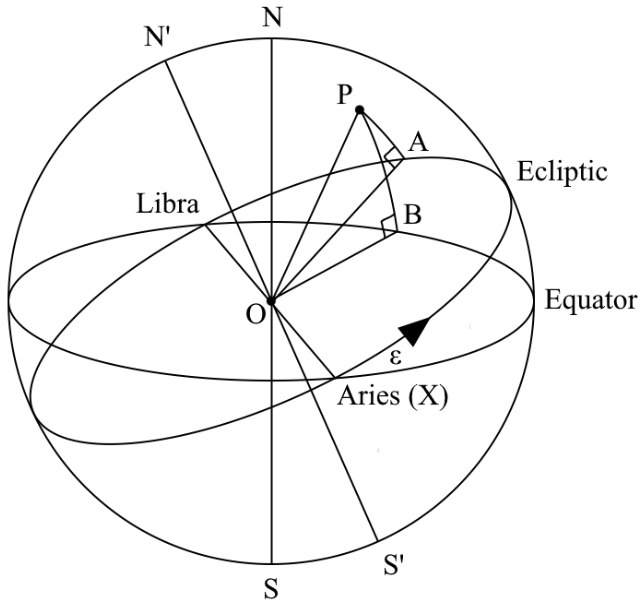
- The Ecliptic Longitude is measured in the plane of the Ecliptic from the Cusp of Aries with East as the positive direction. Declination is measured perpendicular from the Ecliptic such that positive is North.

- By definition of the Ecliptic, the Ecliptic Latitude of the Sun is always $0°$.

- The points of $+90°$ and $-90°$ Latitude are called the *North* and *South Ecliptic Poles* respectively. The celestial poles move extremely slowly around the ecliptic poles due to axial precession, which we will ignore.

- The ecliptic longitude is split into twelve equal signs that span $30°$ each, called the *Zodiac signs*: Aries, Taurus, Gemini, Cancer, Leo, Virgo, Libra, Scorpius, Sagittarius, Capricornus, Aquarius, and Pisces. The definition of original Zodiac signs is obviously possible in a worldbuilding setting of course.

All this is summarized in this diagram below:



In this diagram, The Earth is at the center O, and the motion of the Sun is marked with the arrow. N and S are the celestial poles, and thus the Earth rotates around the axis NS. N' and S' are the ecliptic poles. The direction to the cusp of Aries is marked by "Aries (X)".

- ε is the axial tilt of the Earth.

- XOB is the right ascension, and BOP is the declination.

- XOA is the ecliptic longitude, and AOP is the ecliptic latitude.

1.3 Coordinate Transformations

Coordinate Transformations between the three coordinate systems are given via rotation matrices – the equatorial frame is just the ecliptic frame rotated by ε along the x-axis.

- **Ecliptic to Equatorial:**

$$\begin{bmatrix} x_{\text{equatorial}} \\ y_{\text{equatorial}} \\ z_{\text{equatorial}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\varepsilon) & -\sin(\varepsilon) \\ 0 & \sin(\varepsilon) & \cos(\varepsilon) \end{bmatrix} \begin{bmatrix} x_{\text{ecliptic}} \\ y_{\text{ecliptic}} \\ z_{\text{ecliptic}} \end{bmatrix} \tag{1.3}$$

  The x coordinate stays the same as the ecliptic and equatorial coordinate systems have the same x-axis: the direction of Vernal Equinox.

- **Equatorial to Ecliptic:**

$$\begin{bmatrix} x_{\text{ecliptic}} \\ y_{\text{ecliptic}} \\ z_{\text{ecliptic}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\varepsilon) & \sin(\varepsilon) \\ 0 & -\sin(\varepsilon) & \cos(\varepsilon) \end{bmatrix} \begin{bmatrix} x_{\text{equatorial}} \\ y_{\text{equatorial}} \\ z_{\text{equatorial}} \end{bmatrix} \tag{1.4}$$

  Again, the x coordinate stays the same as the ecliptic and equatorial coordinate systems have the same x-axis: the direction of Vernal Equinox.

These cartesian coordinates can be transformed to spherical coordinates by equation 1.2.

Example 1.1 On January 2, 2024, the Moon's right ascension was $11^h\,19^m\,30.12^s$ and its declination was $+07°\,21'\,42.9''$. Calculate its ecliptic coordinates. (Use $\varepsilon = 23.44°$)

Solution
We first convert the sexagesimal notation to degrees:

$$\begin{aligned} \alpha &= 11^h\,19^m\,30.12^s = 169.8755° \\ \delta &= 07°\,21'\;\;42.9''\;\; = 7.3619° \end{aligned}$$

Keeping in mind that 1^h is $360°/24 = 15°$.
We then convert the equatorial coordinates given to equatorial cartesian coordinates ($\rho = 1$ because the celestial sphere is of arbitrary radius) using equation 1.1:

$$\begin{aligned} x &= \cos(\delta)\cos(\alpha) = -0.976313 \\ y &= \cos(\delta)\sin(\alpha) = 0.174339 \\ z &= \sin(\delta) \qquad\quad\; = 0.128136 \end{aligned}$$

Next, we carry out the matrix multiplication (equation 1.4):

$$\begin{aligned} x_{\text{ecliptic}} &= 1 \cdot x + 0 \cdot y \qquad\quad + 0 \cdot z \qquad\quad = -0.976313 \\ y_{\text{ecliptic}} &= 0 \cdot x + \cos(\varepsilon) \cdot y + \sin(\varepsilon) \cdot z = 0.210923 \\ z_{\text{ecliptic}} &= 0 \cdot x - \sin(\varepsilon) \cdot y + \cos(\varepsilon) \cdot z = 0.0482118 \end{aligned}$$

We then convert to spherical coordinates with $\rho = 1$ using equation 1.2.

$$\begin{aligned} \lambda &= \arctan(0.210923, -0.976313) = 167°\,48'\,32.97'' \\ \beta &= \arcsin(0.0482118/1) \qquad\quad\; = \;\; 2°\,45'\,48.24'' \end{aligned}$$

Ecliptic longitude $167°\,48'\,32.97''$ is in between $150°$ and $180°$, therefore the Moon was in the Zodiac sign Virgo this day. Thus the ecliptic longitude could also be expressed as:

$$\lambda = \text{ Virgo } 17°\,48'\,32.97''$$

The above coordinate systems are *geocentric* in nature, and these are the coordinates an ephemeris lists. However, in order to calculate the ephemeris, we must know the real locations of the planets, and since planets orbit the Sun, we need another set of coordinates, the *Heliocentric Ecliptic Coordinates*. These are the same as the Ecliptic coordinates, the x-axis points towards Aries, and the xy-plane is the Earth's orbital plane (the Ecliptic), but it is centered on the Sun. Thus, the Earth in this frame is on the antipode of the Sun in the geocentric frame:

$$\left.\begin{aligned} \lambda_{\text{Geocentric}} \text{ of the Sun } &= \lambda_{\text{Heliocentric}} \text{ of the Earth } \pm 180^\circ \\ \beta_{\text{Geocentric}} \text{ of the Sun } &= -\beta_{\text{Heliocentric}} \text{ of the Earth} \end{aligned}\right\} \quad (1.5)$$

Thus the Heliocentric Ecliptic latitude of the Earth is also always 0°.

Chapter 2

Orbits

Unlike stars that have fixed coordinates, planets orbit the Sun, and therefore move around in the sky. (In fact, this is where the word "planet" comes from.) Let us study the motion and position of the planets. The most accurate way to calculate the motion of the planets is with a numerical integrator of the equations of motion, but this can be extremely difficult to make such that it guarantees accuracy even in the very long run. In this chapter, we will learn of ways to solve for the motion of the planets *without* an integrator.

2.1 Gravitational Laws

All planets in space obey Newton's law of gravitation.
It tells us:

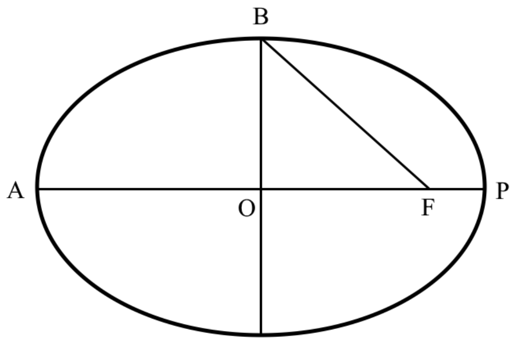
$$\mathbf{F} = -\frac{GMm}{r^3} \cdot \mathbf{r} \tag{2.1}$$

where $\mathbf{F}$ is the gravitational force on the planet, G is the *gravitational constant*, which is $6.674 \cdot 10^{-11}$ m^3 kg^{-1} s^{-2} in SI units, M and m are the masses of the Sun and the planet respectively, and $\mathbf{r}$ and r are the position vector of the planet and its magnitude (the distance between the Sun and the planet) respectively.
Turns out this equation has already been solved for the most part, and the problem of the planets in motion around the Sun, or the Moon in orbit around the Earth, can be modeled as an ideal Newtonian two body problem. Thus they obey Kepler's laws of planetary motion:

1. Planets orbit in an ellipse with the Sun at a *focus*.
    - Thus there is a point in the orbit where it is closest to the Sun and a point where it is furthest away. These two points are diametrically opposite each other.
2. Planets in orbit "sweep out" the same area per unit time.
    - This tells us that the object travels faster when it is in the part of its orbit that is closer to the primary, and slower when it is further away.
3. The square of the orbital period (T) of the planet is proportional to the cube of the *semi-major axis* of the orbit of the planet.

2.2 Kepler's First Law

Since Kepler's first law states that all orbits are ellipses, let us quickly investigate the ellipse.



In the diagram is an ellipse with center O. The distance OP is known as the semi-major axis and is denoted a. (1 Astronomical Unit (AU) is defined by the semi-major axis of the Earth's orbit!) The distance OB is known as the semi-minor axis and is denoted b. The ellipse can be represented algebraically using these two measures: the ellipse is the locus of all points satisfying the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{2.2}$$

Point F is located at the point where the length $BF = a$. This point F is known as the focus of an ellipse and this is where the Sun lies in a planetary orbit.
The points P and A then are the points closest to and farthest away from the focus, and are known as the *periapsis* and *apoapsis* respectively, and *apses* (singular *apsis*) collectively. When the primary object (the object at the focus) is the Sun, these points can be called the *perihelion* and *aphelion*, and when the primary object is the Earth, they can be called the *perigee* and *apogee*.
The amount of "squishing" of an ellipse is given by the quantity c/a where c is the distance OF. This quantity is known as the *eccentricity* and is denoted e. When the eccentricity is 0, the ellipse becomes a perfect circle, and when the eccentricity is greater than or equal to 1, the ellipse breaks and becomes a parabola or hyperbola.
Because $BF = a$, $c^2 = a^2 - b^2$, and e can also be written:

$$e^2 = \frac{a^2 - b^2}{a^2} = 1 - \frac{b^2}{a^2}. \tag{2.3}$$

The periapsis distance is then

$$\left.\begin{aligned} FP &= a - c = a - ae \\ &= a(1 - e) \end{aligned}\right\} \tag{2.4}$$

and the apoapsis distance is

$$\left.\begin{aligned} FA &= a + c = a + ae \\ &= a(1 + e) \end{aligned}\right\} \tag{2.5}$$

Finding the semi-major axis length given the periapsis and apoapsis distances is trivial:

$$a = \frac{1}{2}(FA + FP) \tag{2.6}$$

Example 2.1 Given that the semi-major axis of the orbit of the Earth is 149 598 023 km, and its eccentricity is 0.0167, find the semi-minor axis, the perihelion distance, and the aphelion distance.

Solution
By rearranging equation 2.3:

$$\begin{aligned} b &= \sqrt{a^2(1 - e^2)} \\ &= \sqrt{149\,598\,023^2\ \text{km}^2(1 - 0.0167^2)} \\ &= 149\,577\,161\ \text{km} \end{aligned}$$

The periapsis distance is given by equation 2.4:

$$149\,598\,023\ \text{km}\ (1 - 0.0167) = 147\,099\,736\ \text{km}$$

And the apoapsis distance is given by equation 2.5:

$$149\,598\,023\ \text{km}\ (1 + 0.0167) = 152\,096\,309\ \text{km}$$

Furthermore, the ellipse can also be written in polar form with the focus at the origin: By equation 2.2:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

But remember we have put the (right) focus as the origin and therefore x becomes $x + ae$:

$$\frac{(x+ae)^2}{a^2} + \frac{y^2}{b^2} = 1$$

Substituting $b^2 = a^2(1-e^2)$:

$$\begin{aligned}\frac{(x+ae)^2}{a^2} + \frac{y^2}{a^2(1-e^2)} &= 1\\ (x+ae)^2 + \frac{y^2}{1-e^2} &= a^2\\ x^2 + 2aex + a^2e^2 + \frac{y^2}{1-e^2} &= a^2\end{aligned}$$

Now we substitute $r\cos(\theta)$ and $r\sin(\theta)$ for x and y:

$$(r\cos(\theta))^2 + 2ae(r\cos(\theta)) + a^2e^2 + \frac{(r\sin(\theta))^2}{1-e^2} = a^2$$

Which gives the following quadratic equation in r:

$$\frac{1-e^2\cos^2(\theta)}{1-e^2}r^2 + 2ae\cos(\theta)r - a^2(1-e^2) = 0$$

Finally, solving for r gives us

$$\begin{aligned} r &= \frac{-2ae\cos(\theta) + \sqrt{4a^2e^2\cos^2(\theta) + 4a^2(1-e^2\cos^2(\theta))}}{2\frac{1-e^2\cos^2(\theta)}{1-e^2}}\\ &= \frac{1-e^2(-2ae\cos(\theta) + \sqrt{4a^2e^2\cos^2(\theta) + 4a^2 - 4a^2e^2\cos^2(\theta)})}{2-2e^2\cos^2(\theta)}\\ &= \frac{1-e^2(-2ae\cos(\theta)+2a)}{2-2e^2\cos^2(\theta)}\\ &= \frac{a(1-e^2)(-e\cos(\theta)+1)}{1-e^2\cos^2(\theta)}\\ &= \frac{a(1-e^2)}{1+e\cos(\theta)} \end{aligned} \tag{2.7}$$

Proof of Kepler's First Law

Newton's law of gravitation (equation 2.1) tells us:

$$\mathbf{F} = -\frac{GMm}{r^3} \cdot \mathbf{r}$$

This can be rewritten:

$$\mathbf{F} = -\frac{GMm}{r^2} \cdot \mathbf{u}$$

where $\mathbf{u}$ is the unit vector in the direction of $\mathbf{r}$. Furthermore, Newton's second law of motion tells us:

$$\mathbf{F} = m\mathbf{a}$$

where $\mathbf{F}$ is the force on an object, m is the mass of the object, and $\mathbf{a}$ is its acceleration, defined by $\mathbf{a} = d\mathbf{v}/dt = d^2\mathbf{r}/dt^2$, where $\mathbf{v}$ is the velocity of the object.

By equating the two laws we obtain:

$$\mathbf{a} = -\frac{GM}{r^3} \cdot \mathbf{r}$$

which shows that $\mathbf{a}$ and $\mathbf{r}$ are a scalar multiple of each other (they are parallel), which means that $\mathbf{a} \times \mathbf{r} = \mathbf{0}$. In addition,

$$\begin{aligned} \frac{d}{dt}(\mathbf{r} \times \mathbf{v}) &= \mathbf{r}' \times \mathbf{v} + \mathbf{r} \times \mathbf{v}' \\ &= \mathbf{v} \times \mathbf{v} + \mathbf{r} \times \mathbf{a} \\ &= \mathbf{0} \times \mathbf{0} \\ &= \mathbf{0}. \end{aligned}$$

By integrating both sides of this equation we obtain:

$$\mathbf{r} \times \mathbf{v} = \mathbf{h}$$

where $\mathbf{h}$ is a constant vector. We can reasonably assume $\mathbf{r}$ and $\mathbf{v}$ are not parallel (that is, the planet does not move in a straight line), and thus $\mathbf{h} \neq \mathbf{0}$. Thus all possible $\mathbf{r}$ is perpendicular to $\mathbf{h}$, and since $\mathbf{h}$ is constant, all $\mathbf{r}$ must lie flat on a plane with $\mathbf{h}$ as its normal vector.
Let us now rewrite $\mathbf{h}$.

$$\begin{aligned} \mathbf{h} &= \mathbf{r} \times \mathbf{v} = \mathbf{r} \times \mathbf{r}' = r\mathbf{u} \times (r\mathbf{u})' \\ &= r\mathbf{u} \times (r'\mathbf{u} + r\mathbf{u}') = r^2(\mathbf{u} \times \mathbf{u}') + rr'(\mathbf{u} \times \mathbf{u}) \\ &= r^2(\mathbf{u} \times \mathbf{u}'). \end{aligned}$$

Then,

$$\begin{aligned} \mathbf{a} \times \mathbf{h} &= -\frac{GM}{r^2}\mathbf{u} \times r^2(\mathbf{u} \times \mathbf{u}') \\ &= -GM\mathbf{u} \times (\mathbf{u} \times \mathbf{u}') \\ &= -GM[(\mathbf{u} \cdot \mathbf{u}')\mathbf{u} - (\mathbf{u} \cdot \mathbf{u})\mathbf{u}')] \end{aligned}$$

But since $\mathbf{u}$ is a unit vector ($|\mathbf{u}| = 1$), $\mathbf{u} \cdot \mathbf{u} = 1$ (a constant) and so:

$$\begin{aligned} \frac{d}{dt}(\mathbf{u} \cdot \mathbf{u}) &= \mathbf{u}' \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{u}' \\ &= 2\mathbf{u} \cdot \mathbf{u}' = 0. \\ \therefore \mathbf{u} \cdot \mathbf{u}' &= 0 \\ \therefore \mathbf{a} \times \mathbf{h} &= GM\mathbf{u}' \end{aligned}$$

Therefore we can write

$$\begin{aligned} \frac{d}{dt}(\mathbf{v} \times \mathbf{h}) &= \mathbf{v}' \times \mathbf{h} + \mathbf{v} \times \mathbf{h}' = \mathbf{v}' \times \mathbf{h} \\ &= \mathbf{a} \times \mathbf{h} = GM\mathbf{u}' \end{aligned}$$

Integrating both sides of this equation gives us

$$\mathbf{v} \times \mathbf{h} = GM\mathbf{u} + \mathbf{C} \tag{2.8}$$

where $\mathbf{C}$ is a constant vector.

Let us now choose coordinate axes such that positive z-axis lies in the direction of $\mathbf{h}$. Thus the planet moves in the xy-plane.
Now, because $\mathbf{v} \times \mathbf{h}$ and $\mathbf{u}$ are perpendicular to $\mathbf{h}$ (i.e. in the xy-plane), $\mathbf{C}$ must be in the xy-plane as well. Since $\mathbf{C}$ is a constant vector, we choose the positive x-axis to be in the direction of it, and now r and the angle between $\mathbf{C}$ and $\mathbf{r}$ (which we call θ) define $\mathbf{r}$ in polar coordinates.
From equation 2.8 we now have:

$$\begin{aligned}\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) &= \mathbf{r} \cdot (GM\mathbf{u} + \mathbf{C}) \\ &= GM\mathbf{r} \cdot \mathbf{u} + \mathbf{r} \cdot \mathbf{C} \\ &= GMr\mathbf{u} \cdot \mathbf{u} + |\mathbf{r}||\mathbf{C}|\cos(\theta) \\ &= GMr + rc\cos(\theta)\end{aligned}$$

where $c = |\mathbf{C}|$. Now, solving for r,

$$r = \frac{\mathbf{r} \cdot (\mathbf{b} \times \mathbf{h})}{GM + c\cos(\theta)}$$

If we put $c/(GM) = e$ (and thus $GM = c/e$),

$$r = \frac{1}{GM} \cdot \frac{\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h})}{1 + e\cos(\theta)} = \frac{e}{c} \cdot \frac{\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h})}{1 + e\cos(\theta)}$$

But,

$$\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) = (\mathbf{r} \times \mathbf{v}) \cdot \mathbf{h} = \mathbf{h} \cdot \mathbf{h} = h^2$$

where $h = |\mathbf{h}|$.
Thus:

$$r = \frac{h^2 e/c}{1 + e\cos(\theta)}$$

If we now set $p = h^2/c$, we obtain for r:

$$r = \frac{ep}{1 + e\cos(\theta)}$$

This is a perfectly good polar equation for an ellipse, using p, which is the distance from the focus to the *directrix* of the ellipse.
However, p can be written as $p = a/e + ae$, thus:

$$r = \frac{a + ae^2}{1 + e\cos(\theta)}$$

Which is precisely equation 2.7.

2.3 Kepler's Third Law

Let us now calculate the orbital period T of a planet in a circular orbit, given the orbital radius r. Newton's law of gravitation (equation 2.1) tells us:

$$\mathbf{F} = -\frac{GMm}{r^3} \cdot \mathbf{r}$$

For objects to rotate, there must be a force pointing inwards (called the centripetal force), and this force is given as:

$$\mathbf{F} = -\frac{mv^2}{r^2} \cdot \mathbf{r}$$

Where m is the mass of the rotating body, and v is its speed. Equating the two equations,

$$-\frac{mv^2}{r^2} \cdot \mathbf{r} = -\frac{GMm}{r^3} \cdot \mathbf{r}$$
$$v^2 = \frac{GM}{r}$$
$$v = \sqrt{\frac{GM}{r}}$$

Using the fact that time = distance / speed and that the circumference of the orbit is $2\pi r$,

$$T = \frac{2\pi r}{v} = \frac{2\pi r}{\sqrt{GM/r}}$$
$$= \sqrt{\frac{4\pi^2 r^3}{GM}} \tag{2.9}$$

We can see that equation 2.9 is essentially Kepler's third law, which states that $T^2 \propto r^3$.
Turns out, we can generalize equation 2.9 to elliptical orbits without issue!

$$T = \sqrt{\frac{4\pi^2 a^3}{GM}} \tag{2.10}$$

If we have two objects comparable in mass, they will orbit each other about their center of mass, and the period will be:

$$T = \sqrt{\frac{4\pi^2 a^3}{G(M_1 + M_2)}}. \tag{2.11}$$

Where a is the sum of the two semi-major axes.

Note that period is always positive if the motion is prograde, and negative if the motion is retrograde.

Example 2.2 Given that $G = 6.674 \cdot 10^{-11}$ m^3 kg^{-1} s^{-2}, the mass of the Sun $M_S = 1.989 \cdot 10^{30}$ kg, and the semi-major axis of the orbit of the Earth $a = 1.496 \cdot 10^{11}$ m, calculate the orbital period of the Earth.

Solution
By equation 2.10:

$$T = \sqrt{\frac{4\pi^2 a^3}{GM}}$$
$$= \sqrt{\frac{4\pi^2 (1.496 \cdot 10^{11} \text{ m})^3}{(6.674 \cdot 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2})(1.989 \cdot 10^{30} \text{ kg})}}$$
$$= 3.1554897 \cdot 10^7 \text{ s}$$

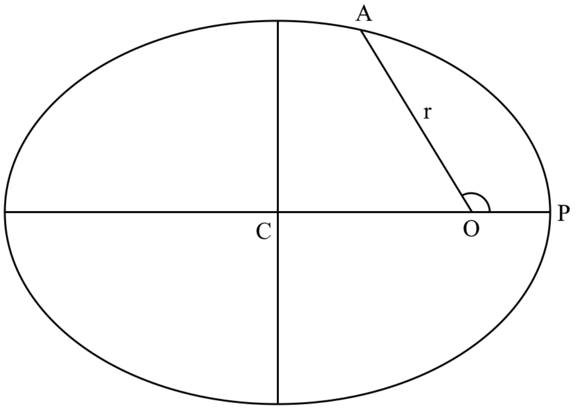
Converting to days:

$$T = 365.219 \text{ dy}$$

Which is close enough to the true value of the year, 365.2422 dy. The difference comes from the fact that there are gravitational perturbations from other Solar System objects on the Earth, and therefore the motion of the Earth is not *exactly* a true two-body problem. The precise math is too difficult for worldbuilding purposes and therefore the perturbation effects of planets on planets will be ignored.

2.4 Perifocal Coordinates

Before we move on, let's come up with a system of describing the position of a planet. A natural way of describing that would be to put the Sun at the origin, and describe its coordinates with the xy-plane as the orbital plane. These coordinates, defined with the positive x axis towards the periapsis, are called the *perifocal coordinates*.



In the diagram, the orbit of a planet A is shown, where O, the origin, is the focus, and therefore the location of the Sun, and P is the perihelion. The angle POA is known as the *true anomaly*, and is denoted ν. This makes it such that the true anomaly is essentially θ in the polar equation for the ellipse. (True anomaly is measured in the direction of the orbit, which in this case we assume is counterclockwise.)

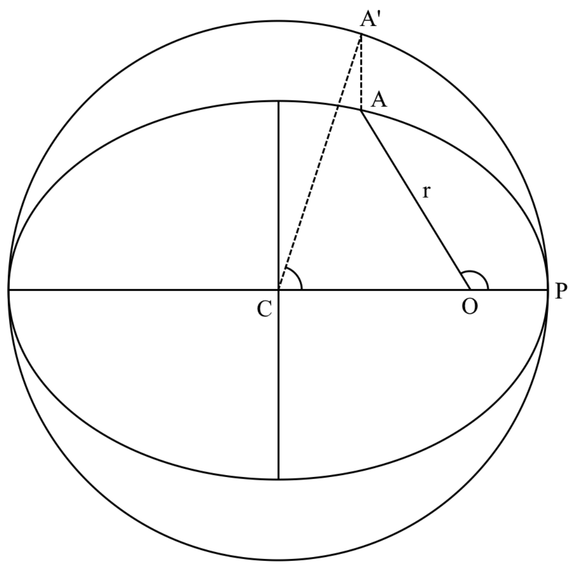
Using the true anomaly, the position (x, y) of the planet can be fully described as:

$$\left.\begin{aligned} x_{\text{perifocal}} &= r\cos(\nu) \\ y_{\text{perifocal}} &= r\sin(\nu) \\ \therefore \nu &= \arctan(y_{\text{perifocal}}, x_{\text{perifocal}}) \end{aligned}\right\} \quad (2.12)$$

Just like in polar coordinates.
Therefore, r here is given by equation 2.7:

$$r = \frac{a(1-e^2)}{1+e\cos(\nu)}. \quad (2.13)$$

Which we can now use to give exact coordinates for x and y.
However, while the true anomaly represents the most intuitive and most physically grounded angle, there is another way to think of things, that will prove to be easier to deal with.



In this diagram, we have drawn a circle with radius a over the ellipse. We then projected the position of the planet A onto this circle and called it A'. Thus a new angle is defined: PCA' defines the *eccentric anomaly*, and is denoted E.

Using E, the coordinates of A will be much easier to determine later. Putting C as the origin,

$$\sin(E) = \frac{x}{a}$$

and because $x^2/a^2 + y^2/b^2 = 1$,

$$\begin{aligned} \left(\frac{y}{b}\right)^2 &= 1 - \left(\frac{x}{a}\right)^2 \\ &= 1 - \sin^2(E) \\ \therefore \frac{y}{b} &= \cos(E) \end{aligned}$$

Therefore,

$$
\begin{aligned}
x &= a\sin(E) \\
y &= b\cos(E)
\end{aligned}
$$

Now, translating the origin to the focus O,

$$
\left.
\begin{aligned}
x_{\text{perifocal}} &= a\sin(E) - ae \\
y_{\text{perifocal}} &= b\cos(E) \\
&= a\sqrt{1-e^2}\sin(E)
\end{aligned}
\right\} \quad (2.14)
$$

Additionally, by the Pythagorean theorem then,

$$
\begin{aligned}
r^2 &= (a\cos(E) - ae)^2 + (a\sqrt{1-e^2}\sin(E))^2 \\
&= a^2\cos^2(E) - 2a^2e\cos(E) + a^2e^2 + a^2(1-e^2)\sin^2(E) \\
&= a^2\cos^2(E) - 2a^2e\cos(E) + a^2e^2 + (a^2 - a^2e^2)(1-\cos^2(E)) \\
&= a^2\cos^2(E) - 2a^2e\cos(E) + a^2e^2 + a^2 - a^2\cos^2(E) - a^2e^2 + a^2e^2\cos^2(E) \\
&= a^2 - 2a^2e\cos(E) + a^2e^2\cos^2(E) \\
&= (a - ae\cos(E))^2 \\
\therefore r &= a(1-e\cos(E)) \qquad (2.15)
\end{aligned}
$$

Let's now relate ν with E. Putting C as the origin again,

$$
\begin{aligned}
\cos(E) &= \frac{x}{a} = \frac{ae + r\cos(\nu)}{a} = e(1-e\cos(E))\cos(\nu) \\
\therefore \cos(E) &= \frac{e+\cos(\nu)}{1+e\cos(\nu)} \qquad (2.16)
\end{aligned}
$$

This formula for E is ambiguous. If $\nu > 180°$, we need to take the negative arccosine value.

The true anomaly can be obtained from the eccentric anomaly by getting $x_{\text{perifocal}}$ and $y_{\text{perifocal}}$ first, then calculating $\arctan(y_{\text{perifocal}}, x_{\text{perifocal})}$ (equation 2.12).

2.5 Kepler's Second Law

We have discussed Kepler's first and third laws. Now let us tackle his second law, which states that the area sweeped out by a planet over a unit time must stay constant.
The area of a sector with radius r and central angle θ is given by:

$$S = \frac{1}{2}r^2\theta$$

And therefore the area sweeped out by a planet over a small increment $d\nu$ of true anomaly is:

$$dS = \frac{1}{2}r^2 d\nu$$

And therefore the area sweeped out per unit time is

$$\frac{dS}{dt} = \frac{r^2}{2}\frac{d\nu}{dt}$$

And this must stay constant.
Because the area of an ellipse is πab,

$$T\frac{dS}{dt} = T\frac{r^2}{2}\frac{d\nu}{dt} = \pi ab$$

where T is the orbital period. Therefore:

$$r^2 \frac{d\nu}{dt} = \frac{2\pi ab}{T}$$

now, denoting $2\pi/T$ as n (this quantity is called the *mean motion*), we obtain:

$$r^2 \nu' = nab = na^2 \sqrt{1-e^2} \tag{2.17}$$

Now, recall equation 2.15:

$$r = a(1 - e\cos(E))$$

Differentiating both sides with respect to time gives us:

$$r' = ae\sin(E)E' \tag{2.18}$$

Now recall equation 2.13:

$$r = \frac{a(1-e^2)}{1+e\cos(\nu)}$$
$$\therefore \frac{1}{r} = \frac{1+e\cos(\nu)}{a(1-e^2)}$$

Differentiating both sides of this equation we obtain:

$$-\frac{r'}{r^2} = -\frac{e\sin(\nu)\nu'}{a(1-e^2)}$$
$$\therefore r' = \frac{e\sin(\nu)r^2\nu'}{a(1-e^2)} \tag{2.19}$$

Now we substitute equation 2.17 in equation 2.19:

$$r' = \frac{nae\sin(\nu)}{\sqrt{1-e^2}}$$

Equating this with equation 2.18:

$$ae\sin(E)E' = \frac{nae\sin(\nu)}{\sqrt{1-e^2}}$$
$$E' = \frac{n\sin(\nu)}{\sqrt{1-e^2}\sin(E)}$$

But $\sqrt{1-e^2}\sin(E)$ is just $y_{\text{perifocal}}/a$ by equation 2.14, and $y_{\text{perifocal}} = r\sin(\nu)$ by equation 2.12, so:

$$E' = \frac{n\sin(\nu)}{r\sin(\nu)/a}$$
$$\therefore rE' = na$$

Now, substituting equation 2.15 for r:

$$(1 - e\cos(E))E' = n$$

Integrating over time on both sides yields:

$$E - e\sin(E) = nt + c.$$

If we measure t from the time of periapsis, then when $t = 0$, $E - e\sin(E) = 0$ since $E = 0$ at periapsis. Therefore $c. = 0$.
Let's also denote nt as M. We call this quantity the *mean anomaly*:

$$M = nt = \frac{2\pi}{T} \cdot t \tag{2.20}$$

Now the equation becomes:

$$M = E - e\sin(E) \tag{2.21}$$

Which is known as **Kepler's Equation**. This equation allows us to relate E and M, thus relating E and t, which allows us to finally calculate the motion of the planets. Keep in mind that we *must use radians* for M and E.

Example 2.3 The orbital period of the Earth $T = 365.2422$ dy, and the eccentricity of its orbit is 0.0167. Additionally, when it is at periapsis, its heliocentric ecliptic longitude is $102° \, 56' \, 49.9''$. Given that the time of perihelion in 2024 was January 3, 2024 $00:38$, calculate the time of Spring Equinox in the Northern Hemisphere in 2024.

Solution

Because ecliptic coordinates are based on the Earth's orbital plane, it effectively is equal to the Earth's perifocal coordinate frame, except that the x-axis is rotated from the direction of perihelion to the direction of Aries. Therefore, we can use our perifocal equations in the ecliptic frame without much trouble.

Spring Equinox in the Northern Hemisphere is defined as $\lambda_{\text{Sun, Geocentric}} = 0°$, therefore (by equation 1.5) it occurs when $\lambda_{\text{Earth, Heliocentric}} = 180°$. Thus, the true anomaly of the Earth at the time of Northern Spring Equinox is:

$$\begin{aligned} \nu &= 180° - 102° \, 56' \, 49.9'' \\ &= 77° \, 3' \, 50.1'' \end{aligned}$$

Now, by equation 2.16 ($\nu < 180°$):

$$\begin{aligned} \cos(E) &= \frac{e + \cos(\nu)}{1 + e\cos(\nu)} \\ &= \frac{0.0167 + \cos(77° \, 3' \, 50.1'')}{1 + 0.0167\cos(77° \, 3' \, 50.1'')} \\ &= 0.239855411 \\ \therefore E &= 1.3285794 \text{ rad} \end{aligned}$$

Now, by Kepler's equation (equation 2.21):

$$\begin{aligned} M &= E - e\sin(E) \\ &= 1.3285794 - 0.0167\sin(1.3285794) \\ &= 1.3123669 \text{ rad} \end{aligned}$$

Therefore, by the definition of M (equation 2.20):

$$\begin{aligned} M &= \frac{2\pi}{T} \cdot t \\ \therefore t &= \frac{TM}{2\pi} \\ &= \frac{365.2422 \text{ dy} \cdot 1.3123669}{2\pi} \\ &= 76 \text{ dy } 6h \, 55m \end{aligned}$$

76 dy $6h$ $55m$ after January 3, 2024 $00:38$ is March 19, 2024 $07:33$.

Comparing to the true time (March 20, 2024 $03:07$), we can see that we are close. The discrepancy comes from rounding error and the fact that the motion of the Earth is not a true two body problem.

However what we really want is the reverse operation of example 2.3: going from a specific time to a location. Sounds easy: M is very easy to calculate, and E gives us the exact coordinates (x, y), and we have a relation between M and E by Kepler's equation. Unfortunately, Kepler's equation

$$M = E - e\sin(E)$$

is transcendental, which means E cannot be solved for M analytically. However, there is hope! We can solve for E *numerically.*

First we rearrange the equation:

$$E - e\sin(E) - M = 0$$

We then define f as a function of E to be $f(E) = E - e\sin(E) - M$. Then we just need to find the root of $f(E) = 0$.
We use the Newton–Raphson method, given by the iterative equation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{2.22}$$

Clearly, we need to find $f'(E)$.

$$\frac{df}{dE} = 1 - e\cos(E)$$

By plugging all the values into equation 2.22, we obtain:

$$E_{n+1} = E_n - \frac{E_n - e\sin(E_n) - M}{1 - e\cos(E_n)} \tag{2.23}$$

which we can use to iteratively obtain better and better approximations of E, which we can then use to find the coordinates of the planet.

Example 2.4 The Earth has orbital period $T = 365.2422$ dy, its orbit has semi-major axis $a = 149.6$ Gm, and its eccentricity $e = 0.0167$. Find the heliocentric ecliptic longitude of the Earth at $t = 76$ dy $6h\,55m$ past periapsis, given that when the Earth at periapsis, its heliocentric ecliptic longitude is $102^\circ\,56'\,49.9''$

Solution
Due to the definition of the ecliptic coordinate frame, we can use our perifocal equations in it without much trouble. (See example 2.3.) We first calculate M by equation 2.20:

$$\begin{aligned} M &= \frac{2\pi}{365.2422 \text{ dy}} \cdot 76 \text{ dy } 6h\,55m \\ &= 1.3123669 \text{ rad} \end{aligned}$$

We now perform the Newton iteration. We first guess $E_1 = M$, and obtain E_2 by equation 2.23.

$$\begin{aligned} E_2 &= M - \frac{M - e\sin(M) - M}{1 - e\cos(M)} \\ &= 1.3285815 \text{ rad} \end{aligned}$$

We perform it again, now using E_2 for E_n.

$$\begin{aligned} E_3 &= E_2 - \frac{E_2 - e\sin(E_2) - M}{1 - e\cos(E_2)} \\ &= 1.3285794 \text{ rad} \end{aligned}$$

Here is the table of repetitions:

n	E_n
1	1.3123669
2	1.3285815
3	1.3285794
4	1.3285794

As we can see, E has quickly converged onto 1.3285794 rad. In general it can be assumed that E_4 or E_5 will be enough. We can now calculate (x, y) with equation 2.14:

$$\begin{aligned} x_{\text{perifocal}} &= a\cos(E) - ae \\ &= 149.6\cos(1.3285794) - 149.6 \cdot 0.0167 \\ &= 33.384 \text{ Gm} \end{aligned}$$

$$\begin{aligned} y_{\text{perifocal}} &= b\sin(E) \\ &= a\sqrt{1-e^2}\sin(E) \\ &= 149.6\sqrt{1-0.0167^2}\sin(1.3285794) \\ &= 145.213 \text{ Gm} \end{aligned}$$

Then, by equation 2.12, the true anomaly is:

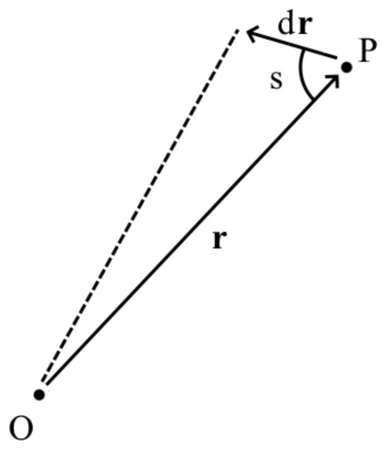
$$\arctan(145.213, 33.384) = 77^\circ\ 3'\ 10.27''$$

Which, when added with the ecliptic longitude of the periapsis $102^\circ\ 56'\ 49.9''$ gives:

$$\lambda_{\text{Earth, Heliocentric}} = 180^\circ\ 17''$$

Which agrees with example 2.3. (The $17''$ is due to stray rounding error.)

Proof of Kepler's Second Law



In this diagram, the motion of a planet P has been depicted. $\mathbf{r}$ is its position vector, and $d\mathbf{r}$ is the small change in position over a small unit time dt. The angle between $\mathbf{r}$ and $d\mathbf{r}$ has been called s.

The area of this triangle dA well represents the small area that is sweeped out by the planet over a small time dt. This area is given by:

$$dA = \frac{1}{2} r dr \sin(s)$$

where r and dr are the magnitudes of $\mathbf{r}$ and $d\mathbf{r}$ respectively. This can be written in vector form as:

$$d\mathbf{A} = \frac{1}{2}\mathbf{r} \times d\mathbf{r}$$

dividing by dt to get the area swept out per unit time,

$$\frac{d\mathbf{A}}{dt} = \frac{1}{2}\mathbf{r} \times \frac{d\mathbf{r}}{dt}$$

Kepler's second law states that this quantity $d\mathbf{A}/dt$ must be constant, which means that $d^2\mathbf{A}/dt^2 = \mathbf{0}$.

Calculating $d^2\mathbf{A}/dt^2 = 0$:

$$
\begin{aligned}
\frac{d^2\mathbf{A}}{dt^2} &= \frac{1}{2}(\frac{d\mathbf{r}}{dt} \times \frac{d\mathbf{r}}{dt} + \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2}) \\
&= \frac{1}{2}(\mathbf{v} \times \mathbf{v} + \mathbf{r} \times \mathbf{a}) \\
&= \frac{1}{2}\mathbf{r} \times \mathbf{a}
\end{aligned}
$$

where $\mathbf{v}$ and $\mathbf{a}$ are the velocity and acceleration of P respectively. However, it has been shown previously (in the proof of Kepler's first law) that $\mathbf{r}$ is parallel to $\mathbf{a}$ and therefore

$$\frac{d^2\mathbf{A}}{dt^2} = \frac{1}{2}\mathbf{r} \times \mathbf{a} = \mathbf{0}.$$

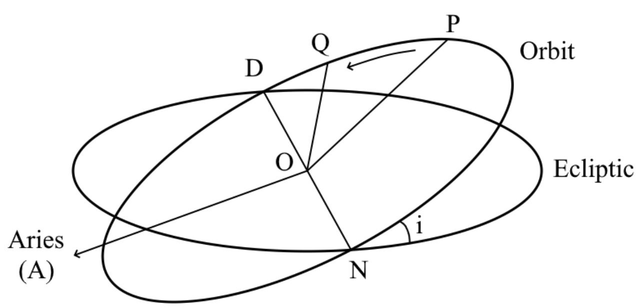
Which proves Kepler's second law.

2.6 Orientation of Orbits in 3D Space

In examples 2.3 and 2.4, we have calculated the position of the Earth in ecliptic coordinates, which was simple because the Earth's perifocal plane was the definition of the Ecliptic plane. Other planets are not so simple. Let:

(p, q, s) denote the perifocal cartesian coordinates of an object, and

(x, y, z) denote the heliocentric ecliptic cartesian coordinates of the object



This diagram depicts the orbit of a non-Earth planet Q where O is the location of the Sun and P is the periaetas of the orbit. The direction of orbit is given by the arrow pointing from P to Q. Thus the true anomaly ν of Q is the angle POQ.

As one can see, the orbit is tilted upwards from the Ecliptic plane by an angle i, and therefore forms a line of intersection. The angle i is called the *inclination*, and the line where the orbit crosses the ecliptic plane (the line ND) is known as the *line of nodes*. The point on the line of nodes where the planet is moving upwards (point N) is called the *ascending node* and the other point is known as the *descending node* (point D).

The angle AON is known as the *longitude of the ascending node* and is denoted Ω, and the angle NOP is known as the *argument of periapsis* and is denoted ω. Then, remembering that the x-axis in the ecliptic frame points towards Aries, and that the p-axis in the perifocal frame points towards the periapsis, the ecliptic frame can be transformed into the perifocal frame by the procedure below:

1. Rotate the ecliptic in the direction of the orbit plane by Ω.
2. Then, rotate it upwards by i.
3. Then, rotate it in the direction of the orbit by ω.

This can be done with rotation matrices:

- - Step 1 - we rotate by Ω about the z-axis.

$$R_1 = \begin{bmatrix} \cos(\Omega) & \sin(\Omega) & 0 \\ -\sin(\Omega) & \cos(\Omega) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- - Step 2 - we rotate by i about the x-axis (which now points in the direction of the ascending node).

$$R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(i) & \sin(i) \\ 0 & -\sin(i) & \cos(i) \end{bmatrix}$$

- - Step 3 - we rotate by ω about the z-axis (which now points perpendicular to the orbital plane).

$$R_3 = \begin{bmatrix} \cos(\omega) & \sin(\omega) & 0 \\ -\sin(\omega) & \cos(\omega) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  Now the x-axis points towards the periapsis.

The full rotation matrix is given by multiplying all these steps together:

$$\begin{bmatrix} p \\ q \\ s \end{bmatrix} = R_3 R_2 R_1 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Actually carrying out the matrix multiplication, this is:

$$\begin{bmatrix} p \\ q \\ s \end{bmatrix} = \begin{bmatrix} \cos\Omega\cos\omega - \sin\Omega\cos i\sin\omega & \sin\Omega\cos\omega + \cos\Omega\cos i\sin\omega & \sin i\sin\omega \\ -\cos\Omega\sin\omega - \sin\Omega\cos i\cos\omega & -\sin\Omega\sin\omega + \cos\Omega\cos i\cos\omega & \sin i\cos\omega \\ \sin\Omega\sin i & -\cos\Omega\sin i & \cos i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{2.24}$$

The reverse transformation, going from the perifocal frame to the ecliptic frame, is given by the transpose of $R_3R_2R_1$:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\Omega\cos\omega - \sin\Omega\cos i\sin\omega & -\cos\Omega\sin\omega - \sin\Omega\cos i\cos\omega & \sin\Omega\sin i \\ \sin\Omega\cos\omega + \cos\Omega\cos i\sin\omega & -\sin\Omega\sin\omega + \cos\Omega\cos i\cos\omega & -\cos\Omega\sin i \\ \sin i\sin\omega & \sin i\cos\omega & \cos i \end{bmatrix} \begin{bmatrix} p \\ q \\ s \end{bmatrix} \tag{2.25}$$

Thus the exact position of Q can be calculated in ecliptic coordinates from the values a, e, i, Ω, and ω and these are called the **orbital elements** of Q.

However, if we recognize the fact that s is almost always going to be 0 in our case (because planets move in the plane of their orbit, there will be almost no cases where we *do* have a s component), we can safely say that, when going from perifocal coordinates to ecliptic coordinates:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\Omega\cos\omega - \sin\Omega\cos i\sin\omega & -\cos\Omega\sin\omega - \sin\Omega\cos i\cos\omega & 0 \\ \sin\Omega\cos\omega + \cos\Omega\cos i\sin\omega & -\sin\Omega\sin\omega + \cos\Omega\cos i\cos\omega & 0 \\ \sin i\sin\omega & \sin i\cos\omega & 0 \end{bmatrix} \begin{bmatrix} p \\ q \\ 0 \end{bmatrix} \tag{2.26}$$

Note that, for the Earth, since $i = 0°$, Ω and ω are on the same plane, and therefore only their sum matters, and as long as Ω and ω sum to the same number, their actual values do not matter and the results of equations 2.24 to 31 will not change.

Example 2.5 Given the orbital elements of Mars and that Mars was last at perihelion on June 21, 2022, calculate its heliocentric ecliptic coordinates on March 19, 2024.

Solution
Mars' orbital elements are given as:

$$\begin{aligned} a &= 227.939 \text{ Gm} \\ e &= 0.0934 \\ i &= 1^\circ\ 51' \\ \Omega &= 47^\circ\ 34'\ 42.7'' \\ \omega &= 286^\circ\ 30' \end{aligned}$$

We first need the mean anomaly of Mars, which involves finding the oribital period T. By equation 2.10:

$$\begin{aligned} T &= \sqrt{\frac{4\pi^2(227.939 \cdot 10^9)^3}{6.67 \cdot 10{-}11 \cdot 1.989 \cdot 10^30}} \\ &= 687 \text{ dy} \end{aligned}$$

Now we follow example 2.4.
Since March 20, 2024 is 637 days after June 21, 2022, by equation 2.20:

$$\begin{aligned} M &= \frac{2\pi}{687 \text{ dy}} \cdot 637 \text{ dy} \\ &= 5.8258938 \text{ rad} \end{aligned}$$

Now we iterate equation 2.23:

n	E_n
1	5.8258938
2	5.7808839
3	5.7809308
4	5.7809308

Thus, by equation 2.14, the perifocal coordinates are:

$$\begin{aligned} p &= 227.939 \cos(5.7809308) - 227.939 \cdot 0.0934 \\ &= 178.499 \text{ Gm} \\ q &= 227.939\sqrt{1 - 0.0934^2} \sin(5.7809308) \\ &= -109.251 \text{ Gm} \\ s &= 0 \text{ Gm} \end{aligned}$$

Since Mars's orbit is not on the ecliptic plane, we must use equation 2.26. We first fill out the matrix:

$$\begin{aligned} \cos\Omega\cos\omega - \sin\Omega\cos i\sin\omega &= 0.913721676036 \\ -\cos\Omega\sin\omega - \sin\Omega\cos i\cos\omega &= 0.405596690451 \\ \sin\Omega\cos\omega + \cos\Omega\cos i\sin\omega &= 0.405159938675 \\ -\sin\Omega\sin\omega + \cos\Omega\cos i\cos\omega &= 0.914006157892 \\ \sin i\sin\omega &= 0.0309535593079 \\ \sin i\cos\omega &= 0.00916886198411 \end{aligned}$$

Thus, by equation 2.26:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0.913721676036 & 0.405596690451 & 0 \\ 0.405159938675 & 0.914006157892 & 0 \\ 0.0309535593079 & 0.00916886198411 & 0 \end{bmatrix} \begin{bmatrix} 178.499 \\ -109.251 \\ 0 \end{bmatrix} = \begin{bmatrix} 118.787 \\ 172.177 \\ 6.527 \end{bmatrix} \text{ [Gm]}$$

In order to get geocentric coordinates, which we will denote by (ξ, η, ζ), we simply subtract the Earth's cartesian coordinates from the target's.

$$\begin{bmatrix} \xi_{\text{Planet}} \\ \eta_{\text{Planet}} \\ \zeta_{\text{Planet}} \end{bmatrix} = \begin{bmatrix} x_{\text{Planet}} \\ y_{\text{Planet}} \\ z_{\text{Planet}} \end{bmatrix} - \begin{bmatrix} x_{\text{Earth}} \\ y_{\text{Earth}} \\ z_{\text{Earth}} \end{bmatrix} \tag{2.27}$$

Considering that $(x_{\text{Sun}}, y_{\text{Sun}}, z_{\text{Sun}}) = (0, 0, 0)$, equation 2.26 proves equation 1.5.
For the Moon, because it orbits the Earth and not the Sun, the perifocal frame is around the Earth, and equation 2.26 would give the geocentric coordinates already, therefore equation 2.27 is not needed.

2.7 Calculating the Ephemeris for a Planet

Let's put it all together.
Algorithm to calculate the ephemeris of a planet:

1. Assign orbital elements to the Earth and the target planet.
2. Choose a time t.
3. Calculate where the planet would be along its orbit at this time using equations 2.20, 2.21, and 2.14.
4. Transform this to heliocentric ecliptic cartesian coordinates by equation 2.26.
5. Repeat step 3 and 4 for the Earth.
6. Calculate the geocentric ecliptic cartesian coordinates of the planet by equation 2.27.
7. Use equations 1.2 and 1.3 to convert to spherical coordinates, either ecliptic or equatorial.
8. Increment t by some amount Δt and repeat steps 3 to 7.

Example 2.5 Calculate Mars' geocentric equatorial coordinates on March 19, 2024. Use $\varepsilon = 23.44°$ for the Earth.

Solution
Recall example 2.4 where we calculated Earth's perifocal coordinates on this date. Using equation 2.26 with

$$\Omega = 0°$$
$$\omega = 102° \, 56' \, 49.9''$$
$$i = 0°$$

(Remember, only the sum $\Omega + \omega$ matters for the Earth) we get for $(x_{\text{Earth}}, y_{\text{Earth}}, z_{\text{Earth}})$:

$$\begin{bmatrix} x_{\text{Earth}} \\ y_{\text{Earth}} \\ z_{\text{Earth}} \end{bmatrix} = \begin{bmatrix} -0.224052873866 & -0.974576990141 & 0 \\ 0.974576990141 & 0.224052873866 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 33.384 \\ 145.213 \\ 0 \end{bmatrix} = \begin{bmatrix} -149.001 \\ 0 \\ 0 \end{bmatrix} \text{ [Gm]}$$

Therefore, using this new result and the result of example 2.5, by equation 2.27:

$$\begin{bmatrix} \xi_{\text{Mars}} \\ \eta_{\text{Mars}} \\ \zeta_{\text{Mars}} \end{bmatrix} = \begin{bmatrix} 118.787 \\ 172.177 \\ 6.527 \end{bmatrix} - \begin{bmatrix} -149.001 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 267.788 \\ -172.177 \\ -6.527 \end{bmatrix} \text{ [Gm]}$$

We need equatorial coordinates, so by equation 1.3:

$$\begin{bmatrix} \xi_{\text{Mars; equatorial}} \\ \eta_{\text{Mars; equatorial}} \\ \zeta_{\text{Mars; equatorial}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(23.44°) & -\sin(23.44°) \\ 0 & \sin(23.44°) & \cos(23.44°) \end{bmatrix} \begin{bmatrix} 267.788 \\ -172.177 \\ -6.527 \end{bmatrix} = \begin{bmatrix} 267.788 \\ -155.372 \\ -74.478 \end{bmatrix} \text{ [Gm]}$$

Then, by equation 1.2:

$$\begin{aligned} \rho &= \sqrt{267.788^2 + (-155.372)^2 + (-74.478)^2} \\ &= 318.430 \text{ Gm} \\ &= 2.128 \text{ AU} \\ \alpha &= \arctan(-155.372, 267.788) \\ &= 21^h\, 59^m\, 30.59^s \\ \delta &= \arcsin\left(\frac{-74.478}{\rho}\right) \\ &= -13°\, 31'\, 34.55'' \end{aligned}$$

Comparing this to the true value of

$$\begin{aligned} \rho &= 2.136 \text{ AU} \\ \alpha &= 21^h\, 59^m\, 47^s \\ \delta &= -13°\, 29'\, 50'' \end{aligned}$$

We can see we were very close considering we ignored all perturbation. Using this method, we can compute an ephemeris for any planet we want in our Solar System. Keep in mind that for the Moon, because it orbits the Earth, equation 2.26 gives the geocentric coordinates directly.

Chapter 3

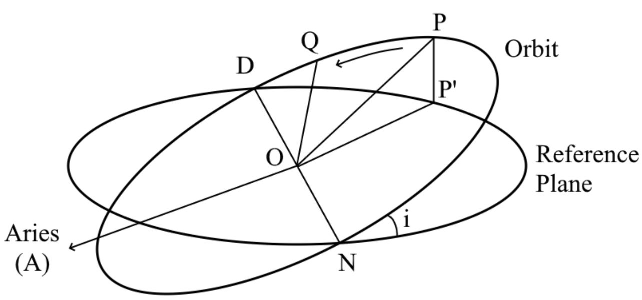
Moons and Orbital Precession

In the previous chapter, we studied the motion of the planets, which was modeled fairly well by Keplerian motion. Moons behave mostly in the same way, but we need to account for a few extra things due to their small size and relatively small semi-major axis lengths. In particular, **orbital precession**, the change in the orbital elements over time caused by gravitational perturbation, cannot be ignored for moons in the same way we have ignored them for the planets.
The two precessions we must account for are:

1. Nodal Precession
    - The change in the location of the node line over time, which is in the opposite direction of the orbit. Hence it is also called the *node regression*.
2. Apsidal Precession
    - The change in the location of the periapsis over time, which is usually in the direction of the orbit. Hence it is also called the *periapsis advance*.

While the orbits of the planets also precess, the rate of precession is so slow (in the order of a few arcminutes of change per century) that they can be ignored. For moons, these rates are much faster. Let us now learn how to calculate these precession rates.

3.1 Change in Ω and ω



This image depicts the orbit of a moon Q around a planet O. We have projected the periapsis of the orbit of the object Q (the point P) down to the reference plane, and thus have created a new point P'. As we have discussed previously, the angle NOP is known as the argument of periapsis and is denoted ω. The angle AOP' (measured in the direction of the orbit) is known as the *longitude of periapsis* and is denoted ϖ (a variant of the letter π).

When discussing apsidal precession, it is usually the precession of the *longitude* of periapsis in question, not the *argument* of periapsis. However, the nodal precession simply refers to the longitude of the ascending node.
These precession rates are not constant in reality but they can be approximated as being more or less constant. Thus, if the nodes precess at a rate $\dot{\Omega}$, the longitude of the ascending node at time t is given as:

$$\begin{aligned}\Omega &= \Omega_0 + \int_{t_0}^{t} \dot{\Omega} dt \\ &= \Omega_0 + (t - t_0)\dot{\Omega} \end{aligned} \tag{3.1}$$

Where Ω_0 is the longitude of the ascending node at time $t = 0$.
If the apses precess at a rate $\dot{\varpi}$, the longitude of periapsis at time t is given as:

$$\begin{aligned} \varpi &= \varpi_0 + \int_{t_0}^{t} \dot{\varpi} dt \\ &= \omega_0 + (t - t_0)\dot{\varpi} \end{aligned} \tag{3.2}$$

Where ϖ_0 is the longitude of the periapsis at time $t = 0$. These make sense. However, in our ephemeris calculations, we have ω and not ϖ. Thus we need a way of calculating ϖ from ω and back. The algorithm below shows how to correctly account for the apsidal precession.

- **Step 1. Calculate ϖ_0 from Ω_0 and ω_0.**

Notice that $\varpi_0 = \Omega_0 + \Lambda_0$, where $\Lambda_0 = NOP'$. Λ then is calculated as follows:
Notice that the periapsis point P is on the orbit and is an angular distance of ω away from the ascending node N. If we define a coordinate frame where O is the origin, the xy-plane is the orbital plane, and the x-axis points towards N, then the spherical coordinates of P are given by $(\omega_0, 0)$.

In a similar coordinate frame, where O is the origin, the x-axis point towards N, but the xy-plane is the reference plane, the coordinates of P are given by (Λ_0, p_0) where p_0 is the angular distance $P'OP$ at time $t = 0$.

To transform from the first set of coordinates to the second set of coordinates, we use a rotation matrix, and thus:

$$\begin{bmatrix} \cos(p_0)\cos(\Lambda_0) \\ \cos(p_0)\sin(\Lambda_0) \\ \sin(p_0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(i) & -\sin(i) \\ 0 & \sin(i) & \cos(i) \end{bmatrix} \begin{bmatrix} \cos(0)\cos(\omega_0) \\ \cos(0)\sin(\omega_0) \\ \sin(0) \end{bmatrix}$$

We only care about Λ, so expanding out the terms for Λ:

$$\begin{aligned} \cos(p_0)\cos(\Lambda_0) &= \cos(0)\cos(\omega_0) = \cos(\omega_0) \\ \cos(p_0)\sin(\Lambda_0) &= \cos(0)\sin(\omega_0)\cos(i) - \sin(0)\sin(i) = \sin(\omega_0)\cos(i) \end{aligned}$$

Thus:

$$\Lambda_0 = \arctan(\sin(\omega_0)\cos(i), \cos(\omega_0))$$

And so finally:

$$\varpi_0 = \Omega_0 + \arctan(\sin(\omega_0)\cos(i), \cos(\omega_0)) \tag{3.3}$$

- **Step 2. Add the precessions.**

We use equations 3.1 and 3.2:

$$\begin{aligned} \Omega &= \Omega_0 + (t - t_0)\dot{\Omega} \\ \varpi &= \varpi_0 + (t - t_0)\dot{\varpi} \end{aligned}$$

- **Step 3. Calculate the new ω.**

We reverse Step 1.
First we find the new Λ:

$$\Lambda = \varpi - \Omega$$

In step 1, we found out that Λ was given by

$$\begin{aligned} \Lambda &= \arctan(\sin(\omega)\cos(i), \cos(\omega)) \\ \therefore \tan(\Lambda) &= \frac{\sin(\omega)\cos(i)}{\cos(\omega)} \\ \therefore \frac{\tan(\Lambda)}{\cos(i)} &= \frac{\sin(\omega)}{\cos(\omega)} \end{aligned}$$

This can be expressed as:

$$\begin{cases} \frac{\sin(\Lambda)}{\cos(i)} = \sin(\omega) \\ \cos(\Lambda) = \cos(\omega) \end{cases}$$

Thus

$$\omega = \arctan\left(\frac{\sin(\Lambda)}{\cos(i)}, \cos(\Lambda)\right)$$

And so finally:

$$\omega = \arctan\left(\frac{\sin(\varpi - \Omega)}{\cos(i)}, \cos(\varpi - \Omega)\right). \tag{3.4}$$

We can now use the new Ω and ω in equation 2.26 to calculate the coordinates of the moon.

Note that if i is very small, then $\cos(i) \approx 1$ and equations 3.3 and 3.4 simplify to:

$$\varpi_0 \approx \Omega_0 + \omega_0 \tag{3.5}$$

$$\omega \approx \varpi - \Omega \tag{3.6}$$

Example 3.1 Given that the Moon has an inclination value of 5.14°, and that the Moon's precession values are: $\dot{\Omega} = -1$ rev/6793 dy and $\dot{\varpi} = 1$ rev/3233 dy. Given that on January 1, 2020, $\Omega = 98°8'24''$ and $\omega = 81°\ 39'$, calcaulate Ω and ω on January 1, 2024.

Solution
We follow the algorithm above.
We first calculate ϖ_0 by equation 3.3:

$$\begin{aligned} \varpi_0 &= \Omega_0 + \arctan(\sin(\omega_0)\cos(i), \cos(\omega_0)) \\ &= 98°\ 8'\ 24'' + \arctan(\sin(81°\ 39')\cos(5.14°), \cos(81°\ 39')) \\ &= 179°\ 45'\ 24.35'' \end{aligned}$$

We now use equations 3.1 and 3.2, where $(t - t_0) = 1461$ dy:

$$\begin{aligned} \Omega &= \Omega_0 + (t - t_0)\dot{\Omega} \\ &= 98°\ 8'\ 24'' + 1461 \cdot \frac{-360°}{6793} \\ &= 20°\ 42'\ 47.65'' \\ \varpi &= \varpi_0 + (t - t_0)\dot{\varpi} \\ &= 179°\ 45'\ 24.35'' + 1461 \cdot \frac{360°}{3233} \\ &= 342°\ 26'\ 29.68'' \end{aligned}$$

We now find the new ω by equation 3.4.

$$\begin{aligned} \omega &= \arctan\left(\frac{\sin(\varpi - \Omega)}{\cos(i)}, \cos(\varpi - \Omega)\right) \\ &= \arctan\left(\frac{\sin(342°\ 26'\ 29.68'' - 20°\ 42'\ 47.65'')}{\cos(5.14°)}, \cos(342°\ 26'\ 29.68'' - 20°\ 42'\ 47.65'')\right) \\ &= 321°\ 36'\ 57.69'' \end{aligned}$$

Thus, on January 1, 2024:

$$\begin{aligned} \Omega &= 20°\ 42'\ 47.65' \\ \omega &= 321°\ 36'\ 57.69'' \end{aligned}$$

Compared to the true values:

$$\Omega = 20^\circ\ 45'$$
$$\omega = 322^\circ\ 46'\ 12''$$

We came close considering we assumed precession is uniform.

Furthermore, we can calculate the rate of change of ω ($\dot{\omega}$) from $\dot{\Omega}$ and $\dot{\varpi}$:
First, let's simplify the expression for ω (this expression introduces some ambiguities but it's okay for this purpose):

$$\omega = \arctan\left(\frac{\tan(\Lambda)}{\cos(i)}\right)$$

Then (reckoning all angles in radians),

$$\dot{\omega} = \frac{\cos^2(i)}{\cos^2(i) + \tan^2(\Lambda)} \cdot \frac{\sec^2(\Lambda)}{\cos(i)} \cdot \dot{\Lambda}$$

But since $\Lambda = \varpi - \Omega$,

$$\dot{\omega} = \frac{\cos(i)\sec^2(\varpi - \Omega)(\dot{\varpi} - \dot{\Omega})}{\cos^2(i) + \tan^2(\varpi - \Omega)}. \tag{3.7}$$

Thus, if one wanted, they could now calculate the new ω by:

$$\omega = \omega_0 + \int_{t_0}^{t} \dot{\omega} dt \tag{3.8}$$

But this is not recommended because $\dot{\omega}$ is a very complex function, unless in cases such as: If i is very small, then $\cos(i) \approx 1$, and:

$$\begin{aligned} \dot{\omega} &\approx \frac{\sec^2(\varpi - \Omega)(\dot{\varpi} - \dot{\Omega})}{1 + \tan^2(\varpi - \Omega)} \\ &= \dot{\varpi} - \dot{\Omega}. \end{aligned} \tag{3.9}$$

Which is consistent with equation 3.6.
In this case where i is very small so $\cos(i) \approx 1$, $\dot{\omega}$ can be approximated as a constant and equation 3.8 may be used.
Similarly, $\dot{\varpi}$ can be calculated from $\dot{\Omega}$ and $\dot{\omega}$:

$$\begin{aligned} \Lambda &= \arctan(\tan(\omega)\cos(i)) \\ \therefore \dot{\Lambda} &= \frac{1}{1 + \tan^2(\omega)\cos^2(i)} \cdot \cos(i)\sec^2(\omega) \cdot \dot{\omega} \end{aligned}$$

But since $\varpi = \Lambda + \Omega$,

$$\dot{\varpi} = \frac{\cos(i)\sec^2(\omega)\dot{\omega}}{1 + \tan^2(\omega)\cos^2(i)} + \dot{\Omega}. \tag{3.10}$$

Note that if i is very small, then $\cos(i) \approx 1$, and:

$$\begin{aligned} \dot{\varpi} &\approx \frac{\sec^2(\omega)\dot{\omega}}{1 + \tan^2(\omega)} + \dot{\Omega} \\ &= \dot{\omega} + \dot{\Omega}. \end{aligned} \tag{3.11}$$

Which is consistent with equation 3.9.

3.2 The Two Types of Moons

This table lists some of the Solar System's most prominent moons:

Name	Primary	a/R	M_M/M_P	T_M/T_P
Moon (Luna)	Earth	60.3	0.01229	0.07480
Io	Jupiter	6.03	0.00005	0.00041
Europa	Jupiter	9.60	0.00003	0.00082
Ganymede	Jupiter	15.3	0.00008	0.00165
Callisto	Jupiter	26.9	0.00006	0.00385
Titan	Saturn	21.0	0.00024	0.00148
Triton	Neptune	14.4	0.00021	0.00010

where a is the semi-major axis of the moon's orbit, R is the primary planet's radius, M_M and M_P are the masses of the moon and planet respectively, and T_M and T_P are the orbital periods of the moon and planet respectively.

As is apparent, our Moon is vastly different from the other moons of our solar system: our moon has much larger a/R, M_M/M_P, and T_M/T_P ratios than other moons. Turns out, these ratios have profound effects on the precession of the orbits of the moons, and thus a distinction must be made between the two kinds of moons: "Luna-type" moons, and "Io-type" moons. (Note: this is not technical terminology.)

- "Luna-type" Moons:
  - Have large a/R, M_M/M_P, and T_M/T_P ratios.
  - Have orbits aligned to the **orbit** of the primary planet. (That is, the inclination to the planetary ecliptic is fixed.)
  - Have most of their precession resulting from the **perturbation from the Sun**.
- "Io-type" Moons:
  - Have small a/R, M_M/M_P, and T_M/T_P ratios.
  - Have orbits aligned to the **equator** of the primary planet. (That is, the inclination to the planetary equator is fixed.)
  - Have most of their precession resulting from the **oblateness of the primary**. (That is, the equatorial bulge of the primary planet.)

Thus it would be more appropriate to call these moons "Orbit-aligned moons" and "Equator-aligned moons" instead.

3.2.1 Orbit-Aligned Moons

Orbit aligned moons have precessions resulting from the gravitational perturbation from the Sun. It can be shown that this perturbation mostly depends on the T_M/T_P ratio, which we denote by m. Solving extremely complex equations of motion leads to these two formulae (derivation too complex to write here):

$$\dot{\Omega} = -\frac{3}{4}m + \frac{9}{32}m^2 + \frac{273}{128}m^3 + \frac{9797}{2048}m^4 + \frac{199273}{24576}m^5 + \frac{6657733}{589825}m^6 \cdots \tag{3.12}$$

$$\begin{aligned}\dot{\varpi} = \frac{3}{4}m + \frac{225}{32}m^2 + \frac{4071}{128}m^3 + \frac{265493}{2048}m^4 + \frac{12822631}{24576}m^5 \\ + \frac{1273925965}{589824}m^6 + \frac{66702631253}{7077888}m^7 + \frac{29726828924189}{679477248}m^8 \cdots\end{aligned} \tag{3.13}$$

These formulae are not exact as they ignore small terms involving e_M^2, e_P^2, i_M^2, and a_M/a_P (not ignoring these would make the formulae unbearably complicated). Thus, for the formula to work best, these parameters for the Moon - Planet system **must be small**.

Equations 3.12 and 3.13 give results in units of rev/T_P.

Example 3.2 Given that for the Moon and the Earth, $T_M = 27.321$ dy and $T_P = 365.242$ dy, Calculate the periods of node regression and perigee advance for the Moon in dy.

Solution
For the Moon and the Earth:

$$\begin{aligned} e_M^2 &= 0.0030 \\ e_P^2 &= 0.0003 \\ i_M^2 &= 0.0080 \text{ rad}^2 \\ a_M/a_P &= 0.0026 \end{aligned}$$

Which are all very small, so equations 3.12 and 3.13 will work. We find that $m = 27.321/365.242 = 0.0748$. Thus, by equations 3.12 and 3.13:

$$\begin{aligned} \dot{\Omega} &= -0.0534631 \text{ rev/yr} \\ \dot{\varpi} &= 0.114575 \text{ rev/yr} \end{aligned}$$

Thus,

$$\begin{aligned} \dot{\Omega} &= \frac{365.242 \text{ dy/yr}}{-0.0534631 \text{ rev/yr}} = -6831.7 \text{ dy/rev} \\ \dot{\varpi} &= \frac{365.242 \text{ dy/yr}}{0.114575 \text{ rev/yr}} = 3187.8 \text{ dy/rev} \end{aligned}$$

These periods of -6831.7 dy and 3187.8 dy do not differ much from their true values of -6793 dy and 3233 dy (the discrepancy comes from ignoring terms involving e_M^2, e_P^2, i^2, and a_M/a_P, and from truncating the infinite formulae at six and eight terms), and they are called the *nodal precession period* and *apsidal precession period* respectively.

There are also more minor perturbations in the lunar orbit by the Sun, called the evection, variation, and the annual equation. Unfortunately these are too complex to get into detail here.

3.2.2 Equator-Aligned Moons

The precession rates of equator aligned moons depend on the gravitational potential field of the primary planet, which can be expressed as an infinite series with coefficients involving *zonal spherical harmonics* $J_1, J_2, J_3, \cdots$. Fortunately, J_2 is about a thousand times larger than all the other terms, so we can focus on the J_2 term and ignore the others. Unfortunately, J_2 is not easily calculable. It depends on the specific distribution of mass in the three-dimensional structure of the primary, and the only way to truly know the value of J_2 is through observation of the precession rates. So what now?

Well, we can make a very crude approximation that the planet is a perfect spheroid of uniform density throughout. Then, J_2 is given by:

$$J_2 \approx \left| \frac{2f}{3} - \frac{R_E^3 w^2}{3GM} \right| \tag{3.14}$$

where f is the flattening, calculated from the equatorial radius R_E and the polar radius R_P of the planet as such:

$$f = \frac{R_E - R_P}{R_E} \tag{3.15}$$

where w is the rotational speed of the planet in rad, G is the gravitational constant, and M is the mass of the planet.

The table of J_2 values by planet in the Solar System is given here:

Name	J_2	Flattening	Equation 3.14
Mercury	$60 \cdot 10^{-6}$	0.000900	$600 \cdot 10^{-6}$
Venus	$4.458 \cdot 10^{-6}$	0.000000	$0.02 \cdot 10^{-6}$
Earth	$1.08263 \cdot 10^{-3}$	0.003353	$1.08136 \cdot 10^{-3}$
Mars	$1.96045 \cdot 10^{-3}$	0.005890	$2.39485 \cdot 10^{-3}$
Jupiter	$14.7360 \cdot 10^{-3}$	0.064870	$13.5152 \cdot 10^{-3}$
Saturn	$16.2980 \cdot 10^{-3}$	0.097960	$12.5902 \cdot 10^{-3}$
Uranus	$3.34343 \cdot 10^{-3}$	0.022930	$5.44115 \cdot 10^{-3}$
Neptune	$3.411 \cdot 10^{-3}$	0.017080	$2.69509 \cdot 10^{-3}$

Evidently, the approximation is very crude; however it's the best we've got. Now that we have J_2, we can calculate the precession rates.
The precession depends on this value which we will call K for simplicity:

$$K = \frac{3J_2 n R_{\text{avg}}^2}{2a^2(1-e^2)^2} \tag{3.16}$$

Where R_{avg} is the average radius of the planet, given by:

$$R_{\text{avg}} = \sqrt[3]{R_E^2 \cdot R_P} \tag{3.17}$$

and a, n, and e are the semi-major axis, mean motion, and eccentricity of the orbit of the satellite respectively.

$$\dot{\Omega} = -K\cos(i) \tag{3.18}$$

where i is the inclination (with respect to the equator) of the orbit of the satellite.
Note that nodal precession is always in the direction opposite to the orbit.

The apsidal precession rate is given as:

$$\dot{\omega} = K\left(2 - \frac{5}{2}\sin^2(i)\right) \tag{3.19}$$

Note that the formula gives $\dot{\omega}$ directly and not $\dot{\varpi}$, and so $\dot{\omega}$ is constant. Thus for equator aligned moons, equation 3.8 gives:

$$\omega = \omega_0 + (t - t_0)\dot{\omega} \tag{3.20}$$

Also note that if $0° \leq i \leq 63.4°$ or $116.6° \leq i \leq 180°$, then the precession of the apses are in the direction of the orbit.

Example 3.3 Mars' largest moon Phobos is an equator aligned moon. Given that for Mars: $J_2 = 1.96045 \cdot 10^{-3}$, $R_{\text{avg}} = 3389.5$ km, and $M = 6.4171 \cdot 10^{23}$ kg, and for Phobos: $a = 9376$ km, $e = 0.0151$, and $i = 1.09°$, find the precession rates of the orbit of Phobos in units of °/dy.

Solution
To find the precession rates, we need K, and so we need T because we need to find $n = 360°/T$

(equation 2.20) for K.
By equation 2.10:

$$\begin{aligned} T &= \sqrt{\frac{4\pi^2 a^3}{GM}} \\ &= \sqrt{\frac{4\pi^2(9376000)^3}{(6.674 \cdot 10^{-11})(6.4171 \cdot 10^{23})}} \\ &= 27564s \\ &= 0.319 \text{ dy} \end{aligned}$$

Now find K by equation 3.16.

$$\begin{aligned} K &= \frac{3J_2 n R_{\text{avg}}^2}{2a^2(1-e^2)^2} \\ &= \frac{3 \cdot (1.96045 \cdot 10^{-3}) \cdot (360°/0.319) \cdot (3389.5)^2}{2 \cdot (9376)^2 \cdot (1 - 0.0151^2)^2} \\ &= 0.4339°/\text{dy} \end{aligned}$$

Then, $\dot{\Omega}$ is given by equation 3.18:

$$\begin{aligned} \dot{\Omega} &= -K\cos(i) \\ &= -0.4339 \cdot \cos(1.09°) \\ &= -0.4338°/\text{dy} \end{aligned}$$

And $\dot{\omega}$ by equation 3.19:

$$\begin{aligned} \dot{\omega} &= K\left(2 - \frac{5}{2}\sin^2(i)\right) \\ &= 0.4339 \cdot \left(2 - \frac{5}{2}\sin^2(1.09°)\right) \\ &= 0.8674°/\text{dy} \end{aligned}$$

Let's check our answers. A data sheet gives values of

$$\begin{aligned} \dot{\Omega} &= -0.4358°/\text{dy} \\ \dot{\varpi} &= 0.4352°/\text{dy} \end{aligned}$$

Since we have $\dot{\omega}$ but the data sheet has $\dot{\varpi}$, we use equation 3.9 with the data sheet values ($i = 1.09°$ is very small):

$$\dot{\omega} \approx 0.4352 - (-0.4358) = 0.8710°/\text{dy}$$

We can see we came pretty close. The discrepancy comes from other perturbation factors, such as the small but not-nonexistent perturbation from the Sun.

3.3 The Anomalistic Period

Remember in Chapter 2 when we calculated the mean anomaly to calculate the eccentric anomaly then to calculate the position of the orbiting object. The mean anomaly was calculated by:

$$M = \frac{2\pi}{T} \cdot t$$

Where t was measured from the time of periapsis. However, when dealing with moons, the periapsis would have moved by the time the moon makes an orbit. Thus, the T used in the formula for M

cannot be just the simple T calculated from 15, which is just the (average) time it takes for the moon to reach the same location (longitude or right ascension) again. (This period is called the *sidereal period.*) To account for the fact that the periapsis has moved, we need a new period, and this is called the *anomalistic period* (as it is the period of repetition of the anomalies). It can be calculated as such:

$$T_A = \frac{T_S T_{\dot{\varpi}}}{T_{\dot{\varpi}} - T_S} \tag{3.21}$$

Where T_S is the sidereal period and $T_{\dot{\varpi}}$ is the period of the precession of periapsis, positive if the precession is in the direction of the orbit, and negative if not.

Example 3.4 Given that the moon has a sidereal period of 27.321 dy, and that the longitude of the perigee precesses by 1 rev every 3233 dy, find the length of the anomalistic period of the Moon.

Solution
By equation 3.21:

$$\begin{aligned} T_A &= \frac{27.321 \cdot 3233}{3233 - 27.321} \\ &= 27.554 \text{ dy} \end{aligned}$$

Which we can now use to calculate the mean anomaly.

3.4 Multi Moon Systems

For multi moon systems, because the moons perturb each other, the math gets increasingly difficult, and the only plausible way to calculate the location of the moons is with a numerical integrator of the equations of motion. For example, in example 3.3, solving for the precession of Deimos would result in less accurate results because of the perturbation of Deimos by Phobos. The only reason example 3.3 worked is because Deimos is much smaller than Phobos and so the effect Deimos has on Phobos is almost negligible.

Recommended solutions are ignoring the perturbation of moons by other moons and just focusing on the perturbations by the equatorial bulge or by the Sun, or just making up precession values for multi moon systems.

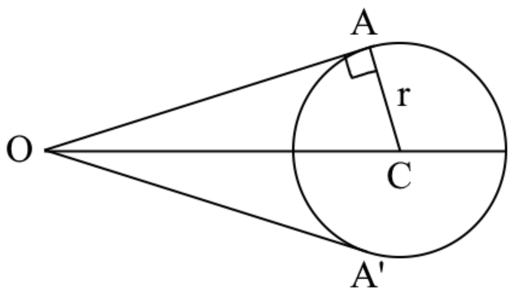
Chapter 4

Geocentric Observation

Let's discuss the effect of viewing objects from the Earth.

4.1 Apparent Size

How large do objects in space appear from our point of view?



Well, in this diagram, an observer at point O is looking at an object with center C and radius r. The *angular size* (also called the *angular diameter*) is then given by the angle AOA', which is $2 \cdot AOC$. The angle AOC, which is called the *angular radius*, is given by this formula:

$$AOC = \arcsin\left(\frac{r}{OC}\right) \tag{4.1}$$

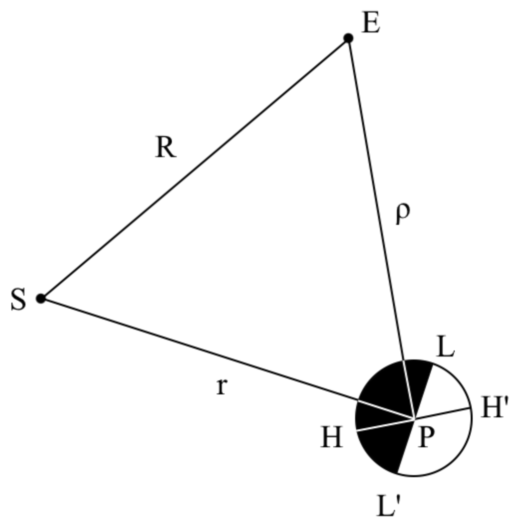
Because AO makes a right angle with AC, the radius. Thus,

$$AOA' = 2\arcsin\left(\frac{r}{OC}\right) \tag{4.2}$$

If the object in question is a disk perpendicular to the viewer and not a sphere, then arctan() instead of arcsin() should be used.

4.2 Phase

Let's now calculate how much of an object would be visible.



In this diagram, S is the position of the Sun, E is the position of the Earth, and P is a celestial body (either a planet or a moon). The line LL' perpendicular to SP and delineates the *terminator line* of P, or the line where night and day are separated. The line HH' is perpendicular to EP and shows the side of the planet visible to E.

The angle SPE is known as the *phase angle*. It can be calculated as follows:

- Step 1: Calculate the angle SEP.

This angle is the angular distance between the Sun and P from the view of the Earth, and can be calculated from the great circle distance formula:

$$SEP = \arccos(\sin(\varphi_S)\sin(\varphi_P) + \cos(\varphi_S)\cos(\varphi_P)\cos(\Delta\vartheta)) \tag{4.3}$$

Where φ_S and φ_P are the declination or latitude of the Sun and P respectively, and $\Delta\vartheta$ is the difference in right ascension or longitude of the Sun and P.

- Step 2: Calculate the distance r.

This is the distance from the Sun to P, and can be found with the law of cosines:

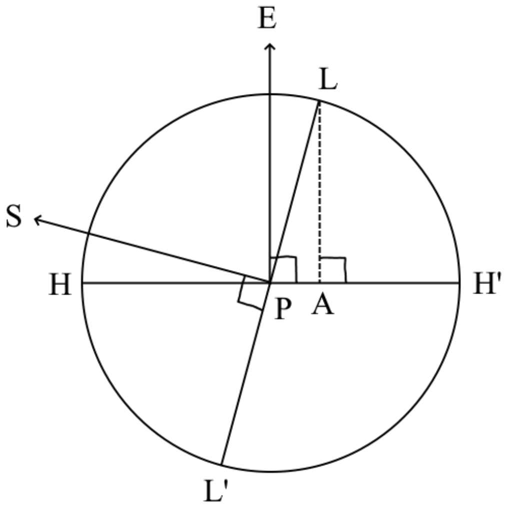
$$r = \sqrt{R^2 + \rho^2 - 2R\rho\cos(SEP)} \tag{4.4}$$

- Step 3: Calculate the phase angle.

This can yet again be done with the law of cosines:

$$\text{Phase Angle } = \arccos\left(\frac{\rho^2 + r^2 - R^2}{2\rho r}\right) \tag{4.5}$$

Note that if all the distances are known already, then steps 1 and 2 can be skipped.

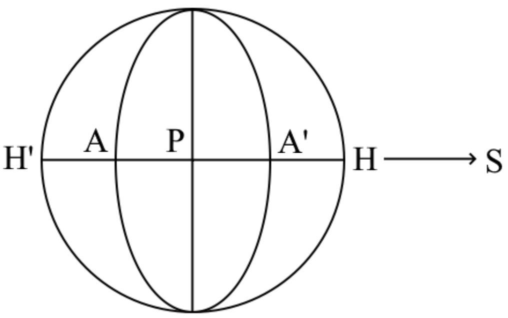


Now in this diagram, we can see that the section of HH' that is illuminated by the Sun is HA, or $HP + PA$. HP is just the planetary radius R_P, and PA is given by $R_P\cos(LPH')$.

But, $LPH' = 90° - EPL$, but also $SPE = 90° - EPL$, so $SPE = LPH'$. Thus the section of HH' illuminated by the Sun is given by $R_P + R_P\cos(SPE)$. Dividing this all by the length of $HH' = 2R_P$, we obtain:

$$\text{Phase } = \frac{1 + \cos(SPE)}{2} \tag{4.6}$$

Where Phase is the fraction of P seen as illuminated from E.



Thus in the view from E, the planet P would look like this diagram, and if we assume the Sun to be shining from the right:

The crescent section between H and A' would be illuminated if the phase was $< 50\%$ and the gibbous section between H and A would be illuminated if the phase was $> 50\%$, with HA'/HH' or HA/HH' being the phase%.

Example 4.1 On January 1, 2024, Venus was at $\alpha = 16^h\,07^m\,26^s$ and $\delta = -18°\,57'\,52''$, and was at a distance 1.1882 AU from the Earth. The Sun was at $\alpha = 18^h\,46^m\,38^s$ and $\delta = -23°\,0'\,10''$, and was at a distance 0.9833 AU from the Earth. Given Venus' radius is $4.045 \cdot 10^{-5}$ AU, calculate the angular size and phase (as a percentage) of Venus on January 1, 2024.

Solution
We get the angular diameter by equation 4.2:

$$\begin{aligned}\text{Angular Diameter } &= 2\arcsin\left(\frac{4.045 \cdot 10^{-5}\text{ AU}}{1.1882\text{ AU}}\right)\\ &= 14''\end{aligned}$$

Now we have to find the phase:
By equation 4.3:

$$\begin{aligned}\text{Sun-Earth-Venus Angle } &= \arccos(\sin(-23^\circ\ 0'\ 10'')\sin(-18^\circ\ 57'\ 52'')+\\ &\cos(-23^\circ\ 0'\ 10'')\cos(-18^\circ\ 57'\ 52'')\cos(18^h\ 46^m\ 38^s - 16^h\ 07^m\ 26^s))\\ &= 37^\circ\ 16'\ 7.87''\end{aligned}$$

Then, by equation 4.4:

$$\begin{aligned}\text{Sun-Venus Distance } &= \sqrt{1.1882^2 + 0.9833^2 - 2 \cdot 1.1882 \cdot 0.9833 \cdot \cos(37^\circ\ 16'\ 7.87'')}\\ &= 0.7205\text{ AU}\end{aligned}$$

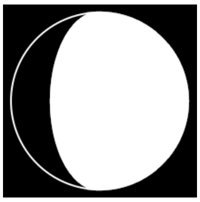
Now, by equation 4.5:

$$\begin{aligned}\text{Phase Angle } &= \arccos\left(\frac{1.1882^2 + 0.7205^2 - 0.9833^2}{2 \cdot 1.1882 \cdot 0.7205}\right)\\ &= 55^\circ\ 43'\ 57.60''\end{aligned}$$

Then finally by equation 4.6:

$$\begin{aligned}\text{Phase (\%)} &= \frac{1+\cos(55^\circ\ 43'\ 57.60'')}{2} \cdot 100\%\\ &= 78.2\%\end{aligned}$$

And thus Venus would look something like:



4.3 Elongation

The *elongation* is the difference in ecliptic longitude between an object and the Sun.

$$\epsilon_{\text{Object}} = \lambda_{\text{Object}} - \lambda_{\text{Sun}} \tag{4.7}$$

This angle is expressed in a range of $(-180^\circ, 180^\circ]$. Thus, if the elongation is positive, the object is *East* of the Sun. This means it will rise after the Sun (because the Earth rotates from West to East, things further East will become visible later), and thus also set after the Sun. This means the object will be *visible in the evening.* If the elongation is negative, the object is *West* of the Sun (Remember that ecliptic longitude is measured with East as positive), and by the same reasoning, will be *visible in the morning.* If the elongation is $\approx 180^\circ$, the object will be visible basically all throughout the night, and if the elongation is $\approx 0^\circ$, it won't be visible at all since it is too close to the Sun.

Example 4.2 Given the data from example 4.1, was Venus a morning star or an evening star on January 1, 2024? Use $\varepsilon = 23.44°$.

Solution
The data from example 4.1:

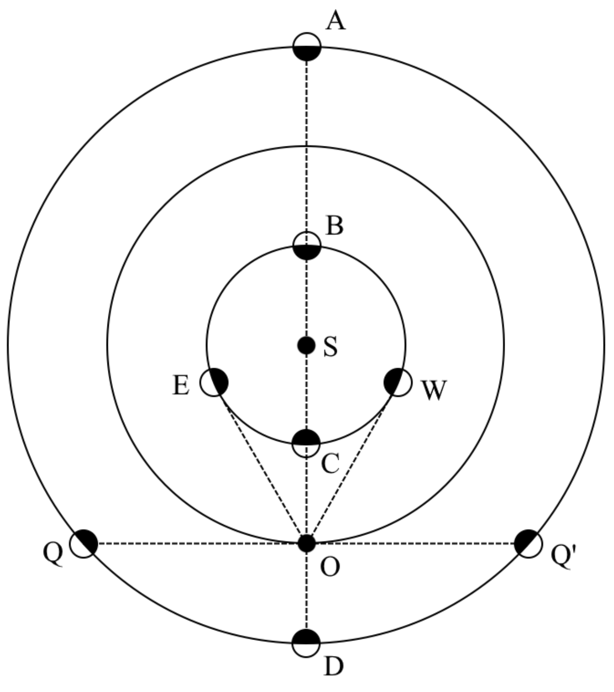
$$\alpha_V = 16^h\,07^m\,26^s$$
$$\delta_V = -18°\,57'\,52''$$

$$\alpha_S = 18^h\,46^m\,38^s$$
$$\delta_S = \delta = -23°\,0'\,10''$$

We need the elongation, so we use equation 1.1, 1.2, 1.4 and equation 4.7:

$$\lambda_V = 243°\,29'\,35.63''$$
$$\lambda_S = 280°\,43'\,11.52''$$
$$\therefore \epsilon = -37°\,13'\,35.89''$$

The elongation is negative, therefore it is a morning star. Additionally, it would be the Eastern side of Venus that would be illuminated (Venus is west of the Sun, therefore its Eastern side faces the Sun). For a more detailed investigation into the lighting direction, see example 6.11.



In this diagram, the orbit of 3 planets are shown with the Sun at the center S. The middle orbit is the orbit of Earth, which is denoted by O. The orientation is such that the direction of West to East is counterclockwise.

Let us focus on the inner planet's orbit. When the inner planet (which we will call I) is at E, its elongation is at its maximum positive value. It is the furthest possible it can be from the Sun. Thus this phenomenon is called "greatest eastern elongation". Similarly, when the inner planet is at W, its elongation is at its minimum value (maximum negative value), and thus this phenomenon is called its "greatest western elongation" Because these points E and W happen at the points where the line OI is tangent to the orbit of I, the phase of I will more or less be 50% at these points. The maximum elongation can be calculated as:

$$\epsilon_{\max} = \arcsin\left(\frac{a_I}{a_O}\right) \tag{4.8}$$

But because orbits are not perfect circles, this formula (and also the fact that the phase of I is 50% at max elongation) is not exact, and the value of this maximum elongation will change depending on the exact orientation of the planets.

Now onto the outer planet. The outer planet cannot have a maximum elongation, its elongation can take any value between $-180°$ and $180°$. Moons that orbit the Earth also cannot have a maximum elongation. When the elongation is 90° (i.e. at the point Q or Q'), the planet is known to be at *quadrature*.

When the elongation of any object is 0°, there is no difference in longitude between it and the Sun, and it is known to be *in conjunction* with the Sun. Note that for inner planets, conjunction can happen at two points in the orbit (at the point B when the phase is $\approx 100\%$ (called the "superior conjunction), and at the point C when the phase is $\approx 0\%$ (called the inferior conjunction)). For outer planets, conjunction can only happen when it is on the far side of the Sun (point A). For moons, conjunction can only happen when the moon is on the near side of the Sun (phase $\approx 0\%$), and this is called the *new moon*.

When the elongation is $180° = -180°$, the object is known to be at *opposition* with the Sun, and can only happen with outer planets (when the outer planet is at D) and moons (in which case this is known as the *full moon*. Note that for outer planets, the phase is $\approx 100\%$ at both opposition and conjunction.

Because the Moon orbits from West to East, its elongation increases as time passes. It goes from being in conjunction to the Sun to being in quadrature, then to being in opposition, then to quadrature again before going back to being in conjunction. It is easy to see that when the Moon's elongation is East, it is *waxing* (its phase is getting larger), and its elongation is West, it is *waning* (its phase is getting smaller).

In addition, if the orbit of a planet has 0° inclination with respect to the ecliptic, the Sun-Earth-Planet angle of equation 4.3 is just equal to the elongation.

4.4 The Synodic Period

The time it takes for the elongation to repeat is called the *synodic period*. This is also the time it takes for the phases to repeat as the phases depend on elongation (the Sun-Earth-Object angle can be approximated as the elongation). Because the elongation depends on the location on both the object and the Earth, the formula involves both periods:

$$T_{P,\ \text{Synodic}} = \left| \frac{T_E T_P}{T_E - T_P} \right| \tag{4.9}$$

where T_E and T_P denote the sidereal period of the Earth and the object respectively.
This formula is not exact as the orbits of the planets are not perfect circles. However it gives the average value.

Evidently, the Earth does not have a synodic orbital period.

Example 4.3 Given their sidereal orbital periods, calculate the synodic period (with respect to Earth) for every planet in the Solar System and the Moon.

Solution
The sidereal periods are given as such:

Name	Sidereal Period
Mercury	87.969 dy
Venus	224.70 dy
Earth	365.24 dy
Moon	27.321 dy
Mars	686.98 dy
Jupiter	4332.6 dy
Saturn	10755 dy
Uranus	30688 dy
Neptune	60195 dy

The synodic orbital period is given by equation 4.9. So for Mercury:

$$\begin{aligned} T_{\text{Mercury, Synodic}} &= \frac{365.24 \cdot 87.969}{365.24 - 87.969} \\ &= 115.88 \text{ dy} \end{aligned}$$

Doing this for all the planets:

Name	Sidereal Period	Equation 4.9	True Value
Mercury	87.969 dy	115.88 dy	115.88 dy
Venus	224.70 dy	583.96 dy	583.92 dy
Earth	365.24 dy	–	–
Moon	27.321 dy	29.530 dy	29.531 dy
Mars	686.98 dy	779.86 dy	779.94 dy
Jupiter	4332.6 dy	398.86 dy	398.88 dy
Saturn	10755 dy	378.08 dy	378.09 dy
Uranus	30688 dy	369.64 dy	369.66 dy
Neptune	60195 dy	367.47 dy	367.49 dy

You can see that the longer the sidereal period is for a planet compared to the Earth's, the closer the synodic period for that planet is to the Earth year. This makes sense because if a planet has a very long orbital period, it is effectively stationary, and then the only factor affecting its elongation is the Earth's orbit. Examples of very far out bodies:

Name	Sidereal Period	Equation 4.9	True Value
Quaoar	105 495 dy	366.51 dy	? dy
500 AU Planet	4 083 507 dy	365.27 dy	? dy

(Note that while Quaoar is a real (dwarf) planet, "500 AU Planet" only serves as a theoretical example.)

Additionally, the synodic period of the Moon (29.53 dy), by definition of the synodic period, is the average time it takes for the Moon to go from new Moon to new Moon, or from full Moon to full Moon. This is known as the lunar month, and is the basis of the length of the month in many calendar systems around the world, such as the Islamic, Jewish, and Chinese calendars.

Applying the synodic period formula for two far-out planets gives the (very) approximate average time between two conjunctions of those two planets.

4.5 Solving for Time of Conjunction

Because the motion of the planets is very complicated, we cannot exactly solve for the time when two bodies are at the same ecliptic longitude. However, we can use numerical methods to get arbitrarily close.

We cannot use the Newton method as we did in chapter 2 because we do not know the derivative of the function we are going to solve for (the difference in longitude between two objects at time t). Thus, we have to use a more crude method: *the bisection method.*

Bisection works as follows:
To find the solution of an equation $f(t) = 0$,

- Step 1. Find two values of t, a and b, such that the sign of $f(a)$ is the opposite of the sign of $f(b)$.
  - If $f(t)$ is continuous between a and b, this guarantees that a solution is in between a and b.
- Step 2. Find the midpoint of a and b, which we will call c.
- Step 3. Find $f(c)$ and compare its sign to $f(a)$ and $f(b)$.
  - If $\text{sign}(f(c)) = \text{sign}(f(a))$, set $a = c$.
  - If $\text{sign}(f(c)) = \text{sign}(f(b))$, set $b = c$.
- Step 4. Repeat steps 2 and 3 until you reach a value of $f(c)$ close enough to 0. Bisection works as follows:

Example 4.4 Find the time of the first new Moon of 2024.

Solution
A Lunar ephemeris can be found online. The one used in this example can be found here:
https://astropixels.com/ephemeris/moon/moon2024.html

We use bisection to solve for when the elongation of the Moon is 0°. We first set $a =$ January 1, 2024 as the first new Moon of 2024 could not have been before this date.

The elongation on this date was $-124.0°$. (We would have to calculate the elongation in a fictional setting as an ephemeris would not be available.) Therefore we need the elongation at time b to be positive.

We set b to be $a + 0.5$ Synodic Period, because then, the elongation would be $\approx -124.0° + 180°$, which is a positive number. (Half of the synodic period of the Moon is called a "fortnight".)
Note that if we subtracted 0.5 Synodic Period instead, the elongation would be $\approx -124.0° - 180° = -304° = 56°$, which is also a positive number, but this does not work as $f(t)$ (the elongation at time t) is not continuous in this region. (There is a jump from $-180°$ to $180°$).

Now we calculate b to be January 16, 2024, and the elongations at a and b were:

t	Date	Elongation
a	January 1, 2024	$-124.0°$
b	January 16, 2024	$+61.8°$

Now we find c, be the midpoint of a and b, to be January 8, 2024.

t	Date	Elongation
a	January 1, 2024	$-124.0°$
b	January 16, 2024	$+61.8°$
c	January 8, 2024	$-45.7°$

The sign of the elongation at time c is the same as the sign at time a, so we set $a = c$:

t	Date	Elongation
a	January 8, 2024	$-45.7°$
b	January 16, 2024	$+61.8°$

Now we repeat the steps: we find the midpoint c and the elongation at c again:

t	Date	Elongation
a	January 8, 2024	$-45.7°$
b	January 16, 2024	$+61.8°$
c	January 12, 2024	$+8.5°$

Now the sign of the elongation at time c is equal to the sign at time b, so we set $b = c$:

t	Date	Elongation
a	January 8, 2024	$-45.7°$
b	January 12, 2024	$+8.5°$

We repeat again. We find c and compare signs of the elongation:
$c =$ January 10, 2024, $\epsilon = -20.6°$, set $a = c$.

t	Date	Elongation
a	January 10, 2024	$-20.6°$
b	January 12, 2024	$+8.5°$

Repeat once more: $c =$ January 11, 2024, $\epsilon = -8.4°$, set $a = c$.

t	Date	Elongation
a	January 11, 2024	$-8.4°$
b	January 12, 2024	$+8.5°$

The ephemeris does not go into hourly detail so we must stop here and say the new Moon happened at some time in between January 11, 2024 and January 12, 2024, i.e. some time during the day of January 11, 2024, but we can go one step into the blind and calculate c one more time for some extra precision: $c =$ January 11, 2024 at $12:00$. (We are remarkably close, the new Moon was on January 11, 2024 at $11:59$.)

In a worldbuilding setting, we can calculate the elongation at times a, b, and c at any detail we want using the methods from chapters 2 and 3 (we would not have to round to the nearest day, because we would not be using a pre-made ephemeris). Thus we can repeat this process to arbitrary precision.

We could have just simply calculated the elongation at even time intervals starting from January 1, 2024, but that is very inefficient and only guarantees precesion up to the size of the time intervals we are taking. Therefore, bisection is better.

The time of conjunction (or any elongation, not just conjunction) of any two bodies can be calculated this way.

4.6 Brightness

Apparent magnitude (denoted by m) is a logarithmic scale measuring how bright objects look in the sky. It stems from a centuries-old system of categorizing stars, class 1 for the brightest stars and class 6 for the dimmest. Thus, our modern system of apparent magnitude also operates with *smaller magnitudes being brighter*: magnitude is defined such that a decrease in apparent magnitude by x is a $100^{x/5}$ times increase in brightness.

For stars, as they produce their own unchanging light, their apparent magnitude is easily calculable:

$$m = -2.5 \log_{10}\left(\frac{F}{F_0}\right) \tag{4.10}$$

Where F is the intensity of the starlight at the observer in W/m^2, and F_0 is a constant equaling $2.518 \cdot 10^{-8}$ W/m^2. F can be calculated with this formula:

$$F = \frac{P}{4\pi R^2} \tag{4.11}$$

Where P is the power output of the star in watts, and R is the distance from the star to the observer in meters. This makes sense as $4\pi R^2$ is the surface area of a sphere with radius R, thus F represents the power output P being spread out over that area.

Example 4.5 Given that the Sun has a power output of $3.8 \cdot 10^{26}$ W and the average distance from the Earth to the Sun is 149 600 000 000 m, calculate the average apparent magnitude of the Sun from the Earth.

Solution
By equation 4.11, the intensity of sunlight on the Earth is:

$$F = \frac{3.8 \cdot 10^{26}}{4 \cdot \pi \cdot 149\,600\,000\,000^2} = 1351.17 \text{ W/m}^2$$

Thus, by equation 4.10:

$$m_{\text{Sun}} = -2.5 \log_{10}\left(\frac{1351.17}{2.518 \cdot 10^{-8}}\right) = -26.8$$

Which is very bright (it is a very negative number, and smaller magnitudes are brighter).

The brightness of planets is harder to calculate: it depends on the *absolute* magnitude, which is the apparent magnitude of an object if the observer were exactly 1 AU away from it.

The actual calculation of the absolute magnitude of planets is very difficult, but if we approximate planets to be ideally reflecting solid matt spheres, it can be calculated by:

$$H = m_{\text{Sun}} - 5 \log_{10}\left(\frac{\sqrt{a}r}{d_0}\right) \tag{4.12}$$

Where m_{Sun} is the apparent magnitude of the Sun at 1 AU, a is the geometric albedo of the planet, a value usually between 0 and 1 describing how much light it reflects (i.e. how white the planet is), r is the radius of the planet, and d_0 is the length of 1 AU, approximately $149,600,000,000$ m.

Note that this formula is a great approximation, for example the Moon of Earth has a brighter half and a darker half, so the albedo is not the same depending on which side is visible (i.e. depends on the elongation from the Sun). This is not reflected in the formula, which assumes all bodeis to have uniform albedo.

Then, the apparent magnitude can be calculated by:

$$m = H + 5\log_{10}\left(\frac{d_{PS}d_{PO}}{d_0^2}\right) - 2.5\log_{10}(q(\alpha)) \tag{4.13}$$

Where d_{PS} is the distance from the planet to the Sun in meters, d_{PO} is the distance from the planet to the observer in meters, d_0 is the same d_0 from before, 1 AU, and α is the phase angle of the planet in degrees.

$q(\alpha)$ is known as the *phase integral.* If we assume planets to be ideally diffusely reflecting spheres, it is given as (the derivation is too difficult to go into detail):

$$q(\alpha) = \frac{2}{3}\left(\left(1 - \frac{\alpha}{180^\circ}\right)\cos(\alpha) + \frac{1}{\pi}\sin(\alpha)\right) \tag{4.14}$$

Note that this formula is a very loose approximation, and does not take into account rings and other factors that may change the brightness of a planet.

Example 4.6 Given that the albedo of Venus is 0.69 and that its radius is 6 051 000 m, Calculate the apparent magnitude of Venus on January 1, 2024.
Use 1 AU = 149 600 000 000 m.

Solution
We found out in example 4.1 that:

$$\begin{aligned} \text{Earth Venus Distance} &= 1.1882 \text{ AU} = 177\,754\,720\,000 \text{ m} \\ \text{Sun Venus Distance} &= 0.7205 \text{ AU} = 107\,786\,800\,000 \text{ m} \\ \text{Venus Phase Angle} &= 55^\circ\ 43'\ 57.60'' \end{aligned}$$

We also found out in example 4.4 that the apparent magnitude of the Sun from 1 AU away is −26.8. Thus, by equation 4.12:

$$\begin{aligned} H &= -26.8 - 5\log_{10}\left(\frac{\sqrt{0.69}\cdot 6\,051\,000}{149\,600\,000\,000}\right) \\ &= -4.4316 \end{aligned}$$

And then by equation 4.14:

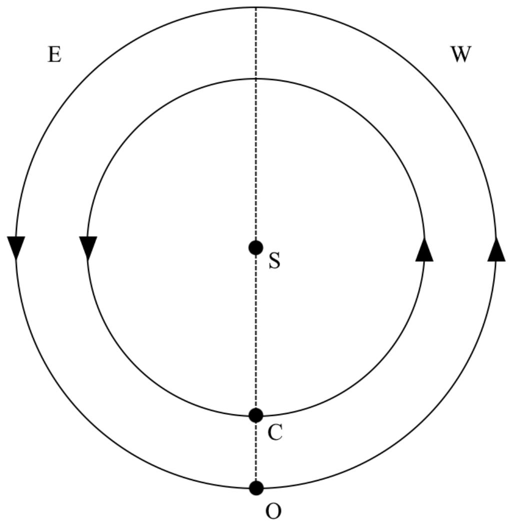
$$\begin{aligned} q(\alpha) &= \frac{2}{3}\left(\left(1 - \frac{55^\circ\ 43'\ 57.60''}{180^\circ}\right)\cos(55^\circ\ 43'\ 57.60'') + \frac{1}{\pi}\sin(55^\circ\ 43'\ 57.60'')\right) \\ &= 0.43452 \end{aligned}$$

And thus finally by equation 4.13:

$$\begin{aligned} m &= -4.4316 + 5\log_{10}\left(\frac{107\,786\,800\,000 \cdot 177\,754\,720\,000}{149\,600\,000\,000^2}\right) - 2.5\log_{10}(0.43452) \\ &= -3.86 \end{aligned}$$

Comparing this to the true value of −4.1, we can see how crude the approximation really is. (We have estimated Venus to be about 15% darker than it really was.) However, this is the best we can realistically do.

4.7 Apparent Retrograde Motion



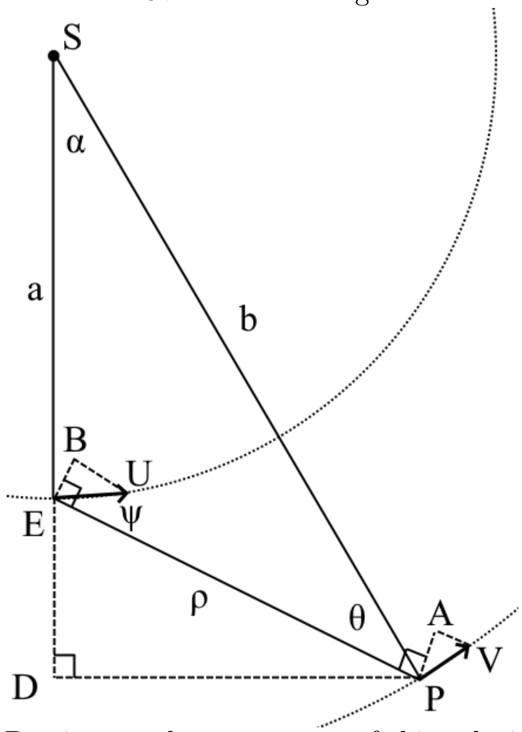
This diagram depicts the Earth (which we have put as stationary at C) and an outer planet, with orbital directions (counterclockwise) marked with arrows.

From the view of the Earth, the outer planet would look to orbit from West to East (i.e. increasing in longitude) most of the time (when it is on the far side of the Sun), but when the planet is near opposition (O), it would appear to move East to West (i.e. decreasing in longitude), because the Earth would be moving faster East than the planet is. This makes it seem like the planet moves backwards for a period of time.

This phenomenon is called *apparent* retrograde motion, as it is not a real physical phenomenon, just an illusion caused by the effects of relative motion. This occurs only once every synodic period, as oppositions only happen once every synodic period.

The true way to calculate when a planet goes into retrograde and for how long it stays in retrograde would be to calculate $d\lambda/dt$ and see when it is less than 0. This is not practical for our purposes because λ is not a simple function at all due to the ellipticity of orbits. Thus one would have to use bisection or similar methods to numerically approximate (to arbitrary precision) when exactly a planet goes into retrograde.

In this section, we will assume for a moment that the orbits of the planets are perfect circles and try to calculate for how long a planet would stay in retrograde (i.e. calculate the average time a planet stays in regrograde). Apparent retrograde motion happens when w_{obs}, the observed angular speed of the outer planet from the view of the Earth, is negative. Because w_{obs} is continuous, we only need to find the times when it is 0 as a continuous function must pass through 0 to go from positive to negative and back. Consider this diagram:



In this diagram, the Sun is at loction S, the Earth at location E, and the outer planet at location P. The distance from the Sun to the Earth is a, the distance from the Sun to the outer planet is b, and the distance from the Earth to the outer planet is denoted ρ.

The velocity of the Earth is shown with the vector $\overrightarrow{EU}$, and the velocity of the outer planet is shown with the vector $\overrightarrow{PV}$. The magnitudes of these vectors will be denoted u and v respectively.

The Earth-Sun-Planet angle is α (not to be confused with right ascension), the Sun-Planet-Earth angle is θ, and the angle UEP is ψ.

First, from the definition of angular velocity, we know that

$$w_{\text{obs}} = \frac{v_{\text{rel}}}{\rho} \tag{4.15}$$

But in v_{rel}, the component of this velocity parallel with EP should not be counted because this part of the velocity is not visible. Thus only the perpendicular components of the velocities $\overrightarrow{EU}$ and $\overrightarrow{PV}$ should be considered:

$$v_{\text{rel}} = |\overrightarrow{PA} - \overrightarrow{EB}| \tag{4.16}$$

It is evident that:

$$\left.\begin{aligned} |\overrightarrow{EB}| &= u\sin(\psi) \\ |\overrightarrow{PA}| &= v\cos(VPA) \end{aligned}\right\} \tag{4.17}$$

Let us determine VPA. Because velocity is always perpendicular to the radius in circular motion (because velocity is always tangential, and tangents to a circle are always perpendicular to radii), angle $SPV = 90° = SEU$, and so

$$\begin{aligned} SPA &= 90° - VPA = 90° - \theta \\ \therefore VPA &= \theta \end{aligned}$$

But $\theta = 180° - \alpha - SEP$, where $SEP = 90° + \psi$. Thus:

$$\begin{aligned} VPA = \theta &= 180° - \alpha - (90° + \psi) \\ &= 90° - (\alpha + \psi) \end{aligned}$$

Thus, equation 4.17 becomes:

$$\left.\begin{aligned} |\overrightarrow{EB}| &= u\sin(\psi) \\ |\overrightarrow{PA}| &= v\cos(90° - (\alpha + \psi)) \\ &= v\sin(\alpha + \psi) \end{aligned}\right\} \tag{4.18}$$

Thus by equation 4.16 and equation 4.15:

$$\begin{aligned} v_{\text{rel}} &= v\sin(\alpha + \psi) - u\sin(\psi) \\ \therefore w_{\text{obs}} &= \frac{v\sin(\alpha + \psi) - u\sin(\psi)}{\rho} \\ &= \frac{v\sin(\alpha)\cos(\psi) + v\cos(\alpha)\sin(\psi) - u\sin(\psi)}{\rho} \end{aligned} \tag{4.19}$$

Now let's analyze the distances and ψ.

$$\begin{aligned} a &= SD - ED \\ &= b\cos(\alpha) - \rho\cos(PED) \end{aligned}$$

But $PED = 90° - \psi$, so:

$$a = b\cos(\alpha) - \rho\sin(\psi)$$

Therefore

$$\sin(\psi) = \frac{b\cos(\alpha) - a}{\rho} \tag{4.20-i}$$

But also,

$$\begin{aligned} PD = b\sin(\alpha) &= \rho\sin(PED) \\ &= \rho\cos(\psi) \\ \therefore \cos(\psi) &= \frac{b\sin(\alpha)}{\rho} \end{aligned} \tag{4.20-ii}$$

Using equations 4.20 we can rewrite equation 4.19 as:

$$\begin{aligned} w_{\text{obs}} &= \frac{v\sin(\alpha)\frac{b\sin(\alpha)}{\rho} + v\cos(\alpha)\frac{b\cos(\alpha)-a}{\rho} - u\frac{b\cos(\alpha)-a}{\rho}}{\rho} \\ &= \frac{vb\sin^2(\alpha) - va\cos(\alpha) + vb\cos^2(\alpha) + ua - ub\cos(\alpha)}{\rho^2} \\ &= \frac{vb + ua - (va + ub)\cos(\alpha)}{\rho^2} \end{aligned}$$

We need to find when $w_{\text{obs}} = 0$, so:

$$\begin{aligned} 0 &= \frac{vb + ua - (va + ub)\cos(\alpha)}{\rho^2} \\ &= vb + ua - (va + ub)\cos(\alpha) \\ \therefore \cos(\alpha) &= \frac{vb + ua}{va + ub} \end{aligned}$$

We can write $v = b \cdot n_P$ and $u = a \cdot n_E$, where n is the mean motion, so:

$$\cos(\alpha) = \frac{a^2 n_E + b^2 n_P}{ab(n_E + n_P)}$$

Since this value for α is the angle when $w_{\text{obs}} = 0$, we need to multiply it by 2 to get the full range of α where there is retrograde motion (the orbit is symmetric about the line of conjunction). Then, dividing 2α by the relative mean motion $n' = n_P - n_E$ gives the time spent in retrograde:

$$T_{\text{retro}} = \left| \frac{2\arccos\left(\frac{a^2 n_E + b^2 n_P}{ab(n_E + n_P)}\right)}{n_P - n_E} \right|$$

Where we use absolute value to keep the answer positive.
Using $n = 360°/T$ gives:

$$T_{\text{retro}} = \left| \frac{2\arccos\left(\frac{a^2 T_P + b^2 T_E}{ab(T_E + T_P)}\right)}{\frac{360°}{T_P} - \frac{360°}{T_E}} \right| \tag{4.21}$$

If we use $b/a = r$, we can simplify this equation even further using kepler's third law because then, $T_P = T_E\sqrt{r^3}$:

$$T_{\text{retro}} = T_E \left| \frac{2\arccos\left(\frac{r + \sqrt{r}}{r\sqrt{r} + 1}\right)}{360° \left(r^{-3/2} - 1\right)} \right| \tag{4.22}$$

Again, this formula is just an approximation assuming the orbits are perfectly circular.

As it turns out, even inner planets follow the same principle and the formula works for inner planets as well. For inner planets, they go retrograde near inferior conjunction (i.e. when the Earth is in opposition with the Sun from the inner planet's POV).

Example 4.7 Given that Venus orbits at 0.7233 AU from the Sun, and the Earth takes 365.24 dy to orbit the Sun, for how long is Venus in retrograde motion from the view of the Earth on average?

Solution
$r = b/a = 0.7233 \text{ AU}/1 \text{ AU} = 0.7233$. Therefore, by equation 4.22:

$$\begin{aligned} T_{\text{retro}} &= 365.24 \text{ dy} \left| \frac{2\arccos\left(\frac{0.7233 + \sqrt{0.7233}}{0.7233\sqrt{0.7233} + 1}\right)}{360° \left(0.7233^{-3/2} - 1\right)} \right| \\ &= 42.2 \text{ dy} \end{aligned}$$

Thus Venus is in retrograde for 42.2 dy every 583.96 dy (See example 4.3), or roughly 7.2% of the time.

Doing this for all the planets yields:

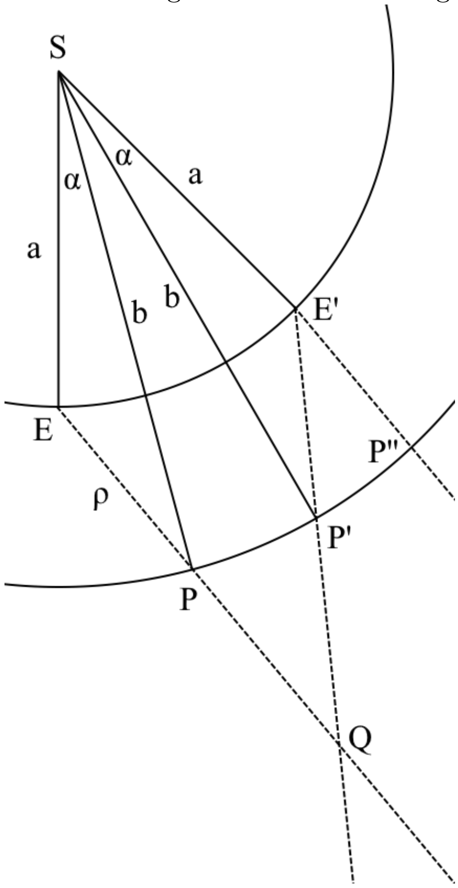
Name	Equation 4.22	True Value	% Retro. Time
Mercury	22.9 dy	21 dy	18%
Venus	42.2 dy	41 dy	7%
Mars	72.7 dy	72 dy	9%
Jupiter	121 dy	121 dy	30%
Saturn	138 dy	138 dy	37%
Uranus	152 dy	151 dy	41%
Neptune	158 dy	158 dy	43%

As we can see, as the planet gets further and further away from the Sun, the time it spends in apparent retrograde motion approaches $T_E/2$ (percentage of time spent in retrograde approaches 50% as the synodic period approaches T_E), which is 182.6 dy for our Earth.
Examples of very far out bodies:

Name	Equation 4.22	True Value	% Retro. Time
Quaoar	163 dy	$\approx$ 163 dy	45%
500 AU Planet	177 dy	? dy	48%

This formula works so well because all these bodies have orbital eccentricities very close to 0.

Now we know how long a planet is in retrograde for, but how *far* does the planet go back by? This is called the angle "recovered" during retrograde.



In this diagram, the Sun is at point S, and E and P are the positions of the Earth and the planet at the moment P starts going into retrograde. E' and P' are the positions of the Earth and the planet at the moment P comes out of retrograde. Thus the angles PSE and $P'SE'$ are both the α that we have calculated before. P'' is located where the planet would be if its geocentric location did not change (i.e. the planet never went into retrograde), that is: EP and $E'P''$ are parallel. (Note that the angle $SE'P''$ is not 180°.) The distances SE and SP (or SE' and SP' since we are assuming circular orbits), and the Earth-planet distance EP are denoted by a, b, and ρ respectively just like before. We are interested in how far the planet has gone backwards in its apparent retrograde motion, that is, we are interested in the angle $P''EP'$.

Because EP and $E'P''$ are parallel, $P''EP' = PQP'$. Because $SEQE'$ forms a quadrilateral, all the angles inside it must sum to 360°. Also note that because triangles $\triangle SEP$ and $\triangle SE'P$ are congruent (by SAS congruency), the angles SEP and $SE'P$ are congruent too. Thus:

$$PQP' = 360° - ESE' - 2SEP \tag{4.23}$$

ESE is just the angular motion the Earth has moved in the time of the planet's retrograde motion:

$$ESE' = \frac{360°}{T_E} \cdot T_{\text{retro}} \tag{4.24}$$

We now only need SEP, which we use the law of sines for:

$$SEP = \arcsin\left(\frac{\sin(\alpha)}{\rho} \cdot b\right) \tag{4.25}$$

ρ can be calculated with the law of cosines:

$$\rho = \sqrt{a^2 + b^2 - 2ab\cos(\alpha)} \tag{4.26}$$

And now we finally have all the necessary parts to calculate PQP'.

If the planet in question is an inner planet (that is, the planet in question is E and the Earth is P), the geometry is the same and all the formulas work out but E and P must be switched:

$$EQE' = 360° - PSP' - 2SPE \tag{4.27}$$

$$PSP' = \frac{360°}{T_P} \cdot T_{\text{retro}} = \frac{360°}{T_E\sqrt{r^3}} \cdot T_{\text{retro}} \tag{4.28}$$

$$SPE = \arcsin\left(\frac{\sin(\alpha)}{\rho} \cdot a\right) \tag{4.29}$$

Where $r = b/a$.
Note that the formula for ρ stays the same.

Note that as with equation 4.22, these formulae are not exact as they assume circular orbits; the angle recovered is different every retrograde cycle.

Example 4.8 Calculate the angle recovered during retrograde for every planet in the Solar System, using $T_E = 365.24$ dy.

Solution
Let's do Uranus as an example of an outer planet:
For Uranus, $b = 19.19$ AU, $\alpha = 73°\ 54'\ 48''$ (the arccosine value of equation 4.22), and $T_{\text{retro}} = 151.8$ dy (by equation 4.22). Thus, by equation 4.26:

$$\begin{aligned}\rho &= \sqrt{1^2 + 19.19^2 - 2 \cdot 1 \cdot 19.19 \cdot \cos(73°\ 54'\ 48'')} \\ &= 18.94 \text{ AU}\end{aligned}$$

Then, by equation 4.25:

$$\begin{aligned}SEP &= \arcsin\left(\frac{\sin(73°\ 54'\ 48'')}{19.19} \cdot 18.94\right) \\ &= 103°\ 10'\end{aligned}$$

Taking the arcsine value that ensures an obtuse angle.
Also, by equation 4.24:

$$\begin{aligned}ESE' &= \frac{360°}{365.24} \cdot 151.8 \\ &= 149°\ 36'\end{aligned}$$

Thus, by equation 4.23:

$$\begin{aligned}PQP' &= 360° - 149°\ 36' - 2 \cdot 103°\ 10' \\ &= 4°\ 4'\end{aligned}$$

Let's now do Venus as an example of an inner planet:
For Venus, $b = 0.7233$ AU, $\alpha = 12° \, 59' \, 48''$ (the arccosine value of equation 4.22), and $T_{\text{retro}} = 42.2$ dy (by equation 4.22).
Thus, by equation 4.26:

$$\begin{aligned} \rho &= \sqrt{1^2 + 0.7233^2 - 2 \cdot 1 \cdot 0.7233 \cdot \cos(12° \, 59' \, 48'')} \\ &= 0.33708 \text{ AU} \end{aligned}$$

Then, by equation 4.29:

$$\begin{aligned} SPE &= \arcsin\left(\frac{\sin(12° \, 59' \, 48'')}{0.33708} \cdot 1\right) \\ &= 138° \, 9' \end{aligned}$$

Also, by equation 4.28:

$$\begin{aligned} PSP' &= \frac{360°}{365.24\sqrt{(0.7233/1)^3}} \cdot 42.2 \\ &= 67° \, 32' \end{aligned}$$

Thus, by equation 4.27:

$$\begin{aligned} EQE' &= 360° - 67° \, 32' - 2 \cdot 138° \, 8' \\ &= 16° \, 12' \end{aligned}$$

Doing this for all the planets yields:

Name	Equations	True Value
Mercury	13° 49′	≈ 9° to 16°
Venus	16° 12′	≈ 15° to 18°
Mars	15° 55′	≈ 10° to 20°
Jupiter	9° 57′	≈ 10°
Saturn	6° 46′	≈ 7°
Uranus	4° 04′	≈ 4°
Neptune	2° 48′	≈ 3°

As we can see, as the planet gets further and further away from the Sun, the average recovered angle during retrogradation approaches 0°.
Examples of very far out bodies:

Name	Equations	True Value
Quaoar	2° 02′	?°
500 AU Planet	0° 13′	?°

Other things regarding apparent retrograde motion:
On Mercury, near perihelion, the orbital speed exceeds the rotational speed, leading to an apparent retrograde motion of the Sun: the Sun appears to go backwards for a period of time, and thus some places on Mercury experience a double sunrise or double sunset.

Sometimes, a conjunction happens between two planets just before one (or both) of them goes into retrograde. This makes it seem like one of the planets goes backwards to conjunct with the other planet again, and when it goes out of retrograde, it goes forward to conjunct again. This phenomenon is called a triple conjunction.

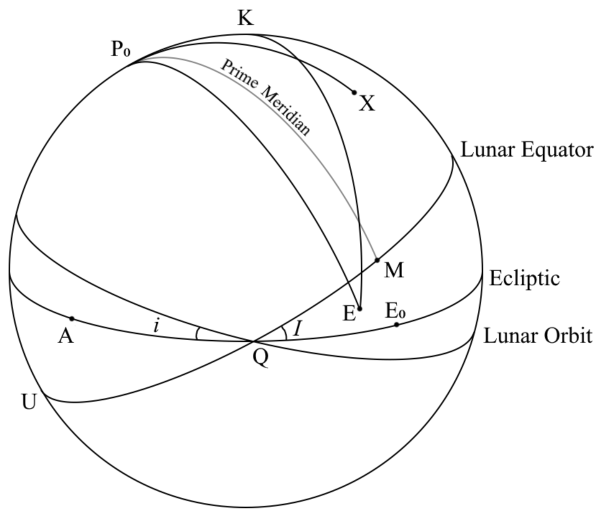
An example is the Jupiter - Saturn triple conjunction of 1980 - 1981:

$$
\begin{aligned}
1^{\text{st}}\text{ Conjunction} &: \text{December 31, 1980}\\
\text{Saturn Retrograde Begins} &: \text{January 18, 1981}\\
\text{Jupiter Retrograde Begins} &: \text{January 24, 1981}\\
2^{\text{nd}}\text{ Conjunction} &: \text{March 4, 1981}\\
\text{Jupiter Retrograde Ends} &: \text{May 27, 1981}\\
\text{Saturn Retrograde Ends} &: \text{June 5, 1981}\\
3^{\text{rd}}\text{ Conjunction} &: \text{July 24, 1981}
\end{aligned}
$$

4.8 Lunar Libration

It is commonly known that only one side of the Moon ever faces the Earth, i.e. the Moon is tidally locked to the Earth. However, this does not mean we can only see exactly 50% of the Moon. Because of the Moon's orbit is not a perfect circle, we can see a bit more than 50% of the Moon depending on the position of the Moon, and this effect is known as the optical libration of the Moon.

The center of the lunar disk as seen from the Earth is evidently the sub-Earth point on the lunar surface. We measure the libration of the Moon with the coordinates of this sub-Earth point. To do this, consider this diagram:



This diagram depicts the Moon. K is the ecliptic pole, i.e. the point with ecliptic latitude of 90°. P_0 is the rotational North pole of the Moon, i.e. the point of the axis of the Moon. Thus the angle of the arc P_0K is the axial tilt of the Moon with regard to the ecliptic, which we denote I. P_0 is not a fixed point on the surface of the Moon however, unlike the true north pole of the Earth, due to the fact that the lunar orbit precesses. This precession of the lunar north pole is in such a way that the ascending node of the lunar equator with respect to the ecliptic, i.e. the point Q, is also the descending node of the orbit of the Moon with respect to the ecliptic. Therefore, if we say that the longitude of the ascending node of the orbit of the Moon is Ω, the ecliptic longitude of the point Q (AQ, where A is the location of Aries) is $\Omega + 180°$. Because of the way the Moon's axis is tilted, it also follows that the angle $KP_0Q = 90°$ and the ecliptic longitude of P_0 is $\Omega + 90°$.

Let us assume that the Moon orbits perfectly uniformly along a circle around the Earth with zero inclination from the ecliptic and calculate this fictional Moon's ecliptic longitude. This is known as the Moon's mean longitude, denoted by L', and is given by:

$$L' = \Omega + \omega + M \tag{4.30}$$

Where ω is the argument of periapsis and M is the mean anomaly. Then, the mean longitude of the Earth from the Moon is $L' + 180°$. We will denote the point with this longitude E_0. Therefore, the arc QE_0 is $180° + L' - (180° + \Omega) = L' - \Omega$. Now, let us define a point called the *mean center of the* *lunar disk*, which we will denote M, that lies on the lunar equator with the same separation from Q as E_0 is from Q. In other words:

$$MQ = L' - \Omega \tag{4.31}$$

M is a fixed point on the lunar surface, and we define the prime meridian of the Moon as the great circle P_0M.

Now we can define the *selenographic* longitude and latitude (l_X, b_X) of any point X on the lunar surface as:

$$\left.\begin{aligned} l_X &= MP_0X \\ b_X &= 90^\circ - P_0X \end{aligned}\right\} \tag{4.32}$$

Now, say the true position of the sub-Earth point is E. If we say that the ecliptic coordinates of the Moon are (λ, β), the ecliptic longitude and latitude of E is $(\lambda + 180^\circ, -\beta)$. Let us say that the selenographic coordinates of the point E is (l, b). Now, using all the quantities we have found so far, we can find that:

$$\left.\begin{aligned} P_0E &= 90^\circ - b \\ KE &= 90^\circ + \beta \\ KP_0E &= 90^\circ - (L' - \Omega) - l = 90^\circ - L' + \Omega - l \\ P_0KE &= \lambda + 180^\circ - (\Omega + 90^\circ) = 90^\circ - (\Omega - \lambda) \\ P_0K &= I \end{aligned}\right\} \tag{4.33}$$

Now, consider the spherical triangle P_0KE. Applying the spherical law of cosines to this triangle and using the results of equations 4.32 gives:

$$\begin{aligned} \cos(P_0E) &= \sin(P_0K)\cos(KE) + \sin(P_0K)\sin(KE)\cos(P_0KE) \\ \therefore \cos(90^\circ - b) &= \cos(I)\cos(90^\circ + \beta) + \sin(I)\sin(90^\circ + \beta)\cos(90^\circ - (\Omega - \lambda)) \\ \therefore \sin(b) &= -\cos(I)\sin(\beta) + \sin(I)\cos(\beta)\sin(\Omega - \lambda) \end{aligned} \tag{4.34-i}$$

Using the first analogue formula for the spherical law of cosines to this triangle with $a = P_0E$ and $B = KP_0E$ gives:

$$\begin{aligned} \sin(P_0E)\cos(KP_0E) &= \cos(KE)\sin(P_0K) - \sin(KE)\cos(P_0K)\cos(P_0KE) \\ \therefore \sin(90^\circ - b)\cos(90^\circ - L' + \Omega - l) &= \cos(90^\circ + \beta)\sin(I) - \sin(90^\circ + \beta)\cos(I)\cos(90^\circ - (\Omega - \lambda)) \\ \therefore \cos(b)\sin(L' - \Omega + l) &= -\sin(\beta)\sin(I) - \cos(\beta)\cos(I)\sin(\Omega - \lambda) \end{aligned} \tag{4.34-ii}$$

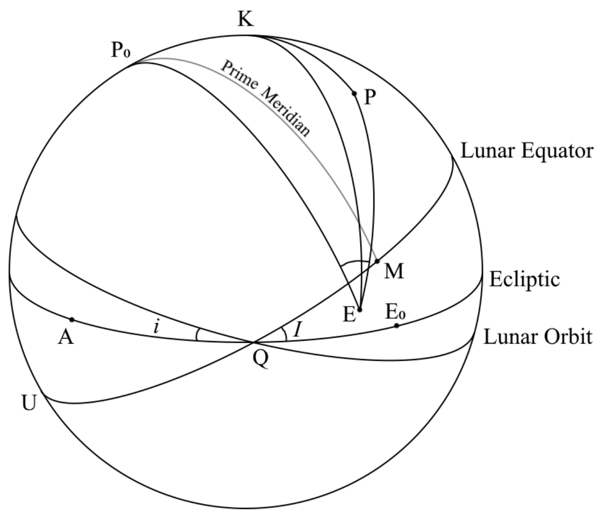
Using the spherical law of sines to this triangle gives:

$$\begin{aligned} \frac{\sin(P_0E)}{\sin(P_0KE)} &= \frac{\sin(KE)}{\sin(KP_0E)} \\ \therefore \frac{\sin(90^\circ - b)}{\sin(90^\circ - (\Omega - \lambda))} &= \frac{\sin(90^\circ + \beta)}{\sin(90^\circ - L' + \Omega - l)} \\ \therefore \sin(90^\circ - b)\sin(90^\circ - L' + \Omega - l) &= \sin(90^\circ + \beta)\sin(90^\circ - (\Omega - \lambda)) \\ \therefore \cos(b)\cos(L' - \Omega + l) &= \cos(\beta)\cos(\Omega - \lambda) \end{aligned} \tag{4.34-iii}$$

To sum up:

$$\left.\begin{aligned} \sin(b) &= -\cos(I)\sin(\beta) + \sin(I)\cos(\beta)\sin(\Omega - \lambda) \\ \cos(b)\sin(L' - \Omega + l) &= -\sin(\beta)\sin(I) - \cos(\beta)\cos(I)\sin(\Omega - \lambda) \\ \cos(b)\cos(L' - \Omega + l) &= \cos(\beta)\cos(\Omega - \lambda) \end{aligned}\right\} \tag{4.34}$$

These equations fully determine l and b and thus fully describe the geocentric optical libration of the Moon.



Another angle of interest however, is the position angle of the axis of rotation of the Moon, i.e. the angle PEP_0, where P is the celestial north pole. This angle shows how tilted the selenographic north pole of the Moon looks from the Earth's perspective, and is denoted c.

First, note that we can separate this angle into two and write $c = PEK + KEP_0$. Also, note that the angle between the ecliptic pole and the celestial pole is the axial tilt of the Earth, so $KP = \varepsilon$. Furthermore, note that angle EKP is the ecliptic longitude of the celestial north pole minus the ecliptic longitude of the point E, i.e. $EKP = 90° - (\lambda + 180°) = -90° - \lambda = 270° - \lambda$.

First let's find PEK. In the triangle PEK, we can use the four parts formula with $a = KE$ and $C = EKP$ to write:

$$\begin{aligned}
\cos(KE)\cos(EKP) &= \sin(KE)\cot(KP) - \sin(EKP)\cot(PEK) \\
\therefore \cos(90° + \beta)\cos(270° - \lambda) &= \sin(90° + \beta)\cot(\varepsilon) - \sin(270° - \lambda)\cot(PEK) \\
\therefore \sin(\beta)\sin(\lambda) &= \cos(\beta)\cot(\varepsilon) + \cos(\lambda)\cot(PEK) \\
\therefore \cot(PEK) &= \frac{\sin(\beta)\sin(\lambda) - \cos(\beta)\cot(\varepsilon)}{\cos(\lambda)} \\
&= \sin(\beta)\tan(\lambda) - \cos(\beta)\cot(\varepsilon)\sec(\lambda) \\
\therefore PEK &= \text{arccot}(\sin(\beta)\tan(\lambda) - \cos(\beta)\cot(\varepsilon)\sec(\lambda)) \qquad \text{(4.35-i)}
\end{aligned}$$

Next, in the triangle KEP_0, since point P_0 has ecliptic longitude $\Omega + 90°$, $EKP_0 = \lambda + 180° - (\Omega + 90°) = \lambda - \Omega + 90°$. Now we can use the four parts formula with $a = KE$ and $C = EKP_0$ to write:

$$\begin{aligned}
\cos(KE)\cos(EKP_0) &= \sin(KE)\cot(KP_0) - \sin(EKP_0)\cot(KEP_0) \\
\therefore \cos(90° + \beta)\cos(\lambda - \Omega + 90°) &= \sin(90° + \beta)\cot(I) - \sin(\lambda - \Omega + 90°)\cot(KEP_0) \\
\therefore \sin(\beta)\sin(\lambda - \Omega) &= \cos(\beta)\cot(I) - \cos(\lambda - \Omega)\cot(KEP_0) \\
\therefore \cot(KEP_0) &= \frac{\sin(\beta)\sin(\lambda - \Omega) - \cos(\beta)\cot(I)}{-\cos(\lambda - \Omega)} \\
&= -\sin(\beta)\tan(\lambda - \Omega) + \cos(\beta)\cot(I)\sec(\lambda - \Omega) \\
&= \sin(\beta)\tan(\Omega - \lambda) + \cos(\beta)\cot(I)\sec(\Omega - \lambda) \\
\therefore KEP_0 &= \text{arccot}(\sin(\beta)\tan(\Omega - \lambda) + \cos(\beta)\cot(I)\sec(\Omega - \lambda)) \qquad \text{(4.35-ii)}
\end{aligned}$$

Therefore,

$$\begin{aligned}
c = {} & \text{arccot}(\sin(\beta)\tan(\lambda) - \cos(\beta)\cot(\varepsilon)\sec(\lambda)) \\
& + \text{arccot}(\sin(\beta)\tan(\Omega - \lambda) + \cos(\beta)\cot(I)\sec(\Omega - \lambda))
\end{aligned} \qquad \text{(4.36)}$$

Together with the apparent size and the phase, the libration angles l, b, and c fully describe the look of the Moon from the Earth.

Example 4.9 Given that, for the Moon, on July 22, 2024 at time 00 : 00, the mean longitude was $L' = 310° \ 29'$, the ecliptic coordinates were $\lambda = 307° \ 21'$ and $\beta = -4° \ 27'$, and the longitude of the ascending node was $\Omega = 8° \ 47'$, find the libration angles l, b, and c.
Use $I = 1.54°$ and $\varepsilon = 23.44°$.

Solution
Equation 4.34 gives:

$$\begin{aligned}\sin(b) &= -\cos(1.54°)\sin(-4° \ 27') + \sin(1.54°)\cos(-4° \ 27')\sin(8° \ 47' - 307° \ 21') \\ &= 0.101093 \\ \therefore b &= \arcsin(0.101093) = 5° \ 48'\end{aligned}$$

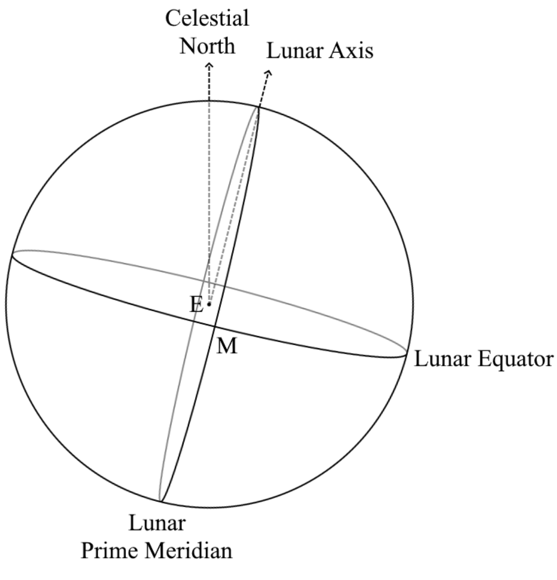
$$\begin{aligned}\cos(b)\sin(L' - \Omega + l) &= -\sin(-4° \ 27')\sin(1.54°) - \cos(-4° \ 27')\cos(1.54°)\sin(8° \ 47' - 307° \ 21') \\ &= -0.873212 \\ \cos(b)\cos(L' - \Omega + l) &= \cos(-4° \ 27')\cos(8° \ 47' - 307° \ 21') \\ &= 0.476739 \\ \therefore L' - \Omega + l &= \arctan(-0.873212, 0.476739) = 298° \ 38' \\ \therefore l &= 298° \ 38' + 8° \ 47' - 310° \ 29' = -3° \ 4'\end{aligned}$$

For c, equations 4.35 give:

$$\begin{aligned}PEK &= \operatorname{arccot}(\sin(-4° \ 27')\tan(307° \ 21') - \cos(-4° \ 27')\cot(23.44°)\sec(307° \ 21')) \\ &= -15° \ 10' \\ KEP_0 &= \operatorname{arccot}(\sin(-4° \ 27')\tan(8° \ 47' - 307° \ 21') + \cos(-4° \ 27')\cot(1.54°)\sec(8° \ 47' - 307° \ 21')) \\ &= 44'\end{aligned}$$

Thus $c = -15° \ 10' + 44' = -14° \ 6'$.

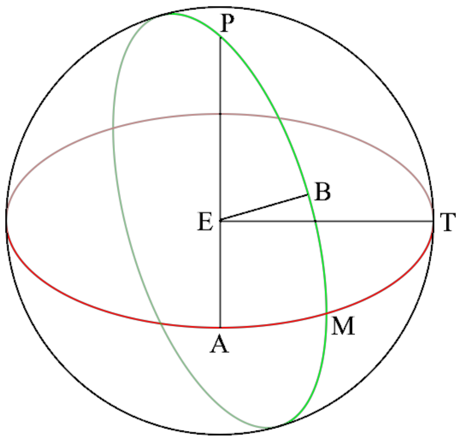
Drawn, this looks like:



The lunar axis is tilted from celestial North by $c = -14° \ 6'$ (negative means clockwise).
The center of the disk of the Moon, E, is at selenographic coordinates $(l, b) = (-3° \ 4', 5° \ 48')$.
North is at the top and East is to the left.

Left: simulated view of the Moon with the calculated libration angles.
Right: simulated view of the Moon from NASA (l, b, $c = -3° \ 25'$, $5° \ 49'$, $-14° \ 22'$).
The error is due to differing ephemeris data.
Note that the extreme right side of the NASA image is dark because the Moon was at 99.4% phase. The left image does not take this into account.

Tip on Drawing Spheres
When drawing a sphere with the center of the disk at coordinates (l, b), follow this guide.



This diagram shows a sphere centered at $l = -30°$ and $b = 30°$. E is the center of the disk, and P is the north pole.

- Let the radius of the white circle ET be r.
- The red ellipse, the equatorial circle, has semiminor axis $EA = r \sin(b)$.
- The green ellipse, the meridional circle, has semiminor axis $EB = r \cos(b) \sin(l)$, and is tilted by $TEB = -\arctan(\sin(b) \tan(l))$ (negative = clockwise).
- To account for the position angle of the axis (the libration angle c), tilt this whole diagram by c (negative = clockwise).

Because the two ellipses intersect at four points, and only one of them is the true location of the zero point M, care must be taken to determine the correct intersection point.

If b is positive, the north pole will be visible (as in this case). if b is negative, the north pole will be hidden behind the sphere.

Chapter 5

Time

We will now move on to observations from the ground at specific times and locations, so let us talk about how to measure time.

5.1 Sidereal and Solar time

When measuring time, two types of time must be distinguished:

- **The Mean Solar (or Synodic) Time** (denoted t)
  - This is the time that bases itself off the Sun. This is the time that all of us are used to. The *average* time of noon (when the Sun is at its highest point in the sky) is $12:00$, or $.5$ (solar) days, and the *average* time of midnight (when the Sun is at its lowest point in the sky) is $00:00$, or $.0$ (solar) days. The Solar day is also called the *synodic day.*
- **The Sidereal Time** (denoted Θ)
  - This is the time that bases itself off the rotation of the Earth. Contrary to popular belief, the rotation period of the Earth is not equal to one solar day. It is instead equal to one sidereal day. These two times are different due to the orbit of the Earth around the Sun. One sidereal day after some point in time, the distant stars will return to the same position in the sky, but because the Earth has orbited the sun and has moved in that time period (or, from the Earth's perspective, the Sun has moved), the Sun will have not returned to the same position. Therefore there is a discrepancy between the two times.
  - *Sidereal time measures the amount the Earth has rotated.* Where do we start measuring it then? Well, it is measured with 0 sidereal time being when Aries is highest in the sky. In the Northern hemisphere, this is when Aries is directly South. In the Southern hemisphere, it is when Aries is directly North.
  - We call the North-South line the "Meridian", and this can be extended as a whole circle around us. It is the circle that starts at North, goes straight up to the point right above our head (the "Zenith"), down to direct South, and down further to the point right beneath our feet (the "Nadir") and then back up to North. This is called the "Meridian circle" or "Meridian plane".
  - The sidereal time then measures how far away Aries is from this Meridian circle. Since the Earth rotates from East to West, things rise in the East and set in the West. This means that as time progreses, Aries will go away from the meridian towards West before setting beneath the horizon, and then come back up in the East later. Thus, sidereal time is measured as the distance of Aries from the Meridian with *West as positive.*
  - This further means that sidereal time could also be measured how far East the meridian is from Aries, i.e. the *right ascension of the meridian.*
  - Sidereal time is often measured in degrees.

A further investigation into the difference between the two times: as mentioned earlier, after one sidereal day, the Earth is facing the same point in the sky, and thus the stars will have returned to the same point in the sky. However, because the Earth has orbited an amount around the Sun in that time period, the Sun will not have returned to the same position in the sky.
So how long is a solar day in comparison to a sidereal day? Well, think about it this way:

- Say at time 0, the Prime Meridian of the Earth is facing away from the Sun, and the Sun is exactly at Aries.
- Say n sidereal days after some date, the Earth is at the opposite end of its orbit.
- Then, the Prime Meridian will still be pointing towards Aries, but now it is facing the Sun instead of away. This means that there was 0.5 more sidereal days in that time period than there were Solar days.
- Thus, there is exactly 1 more sidereal day in 1 year than there are Solar days.
- The same argument applies for retrograde rotation, but there is 1 fewer sidereal days than Solar days.

Thus, since there are $Y \pm 1$ sidereal days per Y synodic days, where Y is the length of the year in solar days,

$$\text{Sidereal Day} = \frac{Y}{Y \pm 1} \cdot \text{Solar Day} \tag{5.1}$$

Example 5.1 The Earth has an orbital period of 365.2422 Earth Solar days while Venus has an orbital period of 1.92 Venusian Solar days. Calculate the length of the sidereal day on Earth and Venus, keeping in mind Venus has a retrograde rotation.
(The length of an Earth Solar day is 24 hours while the length of a Venusian Solar day is 116.75 Earth Solar days.)

Solution
We use equation 5.1.
Since Earth has a prograde orbit,

$$\text{Sidereal Day Length} = \frac{Y}{Y+1} \cdot \text{Solar Day Length}$$

Substituting the numbers,

$$\text{Earth Sidereal Day Length} = \frac{365.2422}{365.2422+1} \cdot 24h = 23h\,56m\,4s$$

Since Venus has a retrograde orbit,

$$\text{Sidereal Day Length} = \frac{Y}{Y-1} \cdot \text{Solar Day Length}$$

Substituting the numbers,

$$\text{Venus Sidereal Day Length} = \frac{1.92}{1.92-1} \cdot 116.75\ \text{ dy } = 243\ \text{ dy}$$

Comparing these values to Wikipedia, we can see we are correct.

Due to random fluctuations in the rotation rate of the Earth, the length of the sidereal day fluctuates by a second or two. *We will ignore this for the purposes of worldbuilding.*

5.2 Converting between Times

The prime meridian is the reference longitude on the Earth. This is where longitude is measured from, and it is also where the standard time is measured. All other solar times can be converted to standard time via this formula:

$$\text{Standard Time } (t) = \text{Local Time} - \frac{l/360^\circ}{\text{Solar Day Length}} \tag{5.2}$$

Where l is the local longitude (East is positive).
If the time is given as an angle, the following formula is perfectly viable:

$$\text{Standard Time } (t) = \text{Local Time} - l \tag{5.3}$$

The benefit to worldbuilding is that we can decide when time 0 and when day 0 is. **Here, we define time 0 to be the time of Spring Equnox on the prime meridian, and we shall also, for the sake of convenience, also say that the Spring Equinox happened at exactly midnight.** This means, that at solar time (t) = 0, the sidereal time was exactly -0.5 sidereal days, as Aries (coincident with the Sun) was at midnight. Under this presumption, the conversion from Solar time to sidereal time is very easy. Since the length of a sidereal day is exactly $Y/(Y \pm 1)$ of a solar day, we just multiply the time elapsed, in days, from $t = 0$ by $(Y \pm 1)/Y$ to get the sidereal time, then subtract by 0.5 sidereal days to account for the fact that Aries was at midnight at $t = 0$.

$$\left.\begin{aligned} \Theta &= \frac{Y \pm 1}{Y} \cdot t - 0.5 \\ t &= (\Theta + 0.5) \cdot \frac{Y}{Y \pm 1} \end{aligned}\right\} \tag{5.4}$$

Example 5.2 An observation was made on planet P on solar day 175 at solar time 05 : 16 : 34 at $l = 165^\circ E$. Calculate the standard sidereal time at the time of the observation.
(Assume a year length of 289.42 solar days, a solar day length of 24 hours, and prograde rotation.)

Solution
First, using equation 5.2, we determine the standard time of observation.

$$\begin{aligned} \text{Standard Time } (t) &= 05:16:34 - \frac{165^\circ/360^\circ}{24h} \\ &= 05:16:34 - 11h \\ &= -1 \text{ dy } 18:16:34 \end{aligned}$$

Where dy means Solar days. This means the standard time at the time of observation was solar day 174 at 18 : 16 : 34, or at $t = 174.7615$ days. Then, using equation 5.4:

$$\Theta \text{ (in days)} = \frac{289.42 + 1}{289.42} \cdot 174.7615 - 0.5 = 174.8653$$

Thus the standard sidereal time at the time of measurement was sidereal day 174, $311^\circ\ 31'\ 12.25''$. This can be interpreted as the fact that at the time of measurement, at the prime meridian, the cusp of aries had rotated $311^\circ\ 31'\ 12.25''$ from the meridian, or in other words: **the right ascension of the prime meridian was** $311^\circ\ 31'\ 12.25'' = 20^h\ 46^m\ 4.82^s$.

Furthermore, the local sidereal time can be calculated by equation 5.3:

$$\Theta_{\text{local}} = \Theta_{\text{standard}} + l = 175 \text{ sdy } 116^\circ\ 31'\ 12.25''$$

Where sdy means sidereal days.

To convert from sidereal time to mean solar time, it is harder. Often, from later on calculations that give us the sidereal time of an event, the whole part of the sidereal time will not be apparent. Therefore we must guess by knowing the solar date. However, equation 5.4 still holds.

Example 5.3 On planet P on local solar day 175, an observation was made at $l = 165°E$ at local sidereal time $116°\ 31'\ 12.25''$. Calculate the standard mean solar time.

Solution
We convert to standard sidereal time by subtracting the longitude:

$$\text{Standard } \Theta = 116°\,31'\,12.25'' - 165° = -1 \text{ sdy } 311°\,31'\,12.25'' \qquad \text{(i)}$$

We then find the sidereal day corresponding to solar day 175:

$$\Theta_{t=175.00} = \frac{290.42}{289.42} \cdot 175 - 0.5 = 175 \text{ sdy } 37°\,40'\,36.24''$$

Which we truncate to:

$$\Theta_{t=175.00} = 175 \text{ sdy} \qquad \text{(ii)}$$

Combining the results from (i) and (ii), we try $\Theta = 175$ sdy -1 sdy $311°\ 31'\ 12.25''$. Using equation 5.4 with 0.5 sidereal days = 180°,

$$\begin{aligned} t &= (174 \text{ sdy } 311°\,31'\,12.25'' + 180°) \cdot \frac{289.42}{290.42} \\ &= 174.7615 \text{ dy} \\ &= 174 \text{ dy } 18:16:34 \end{aligned}$$

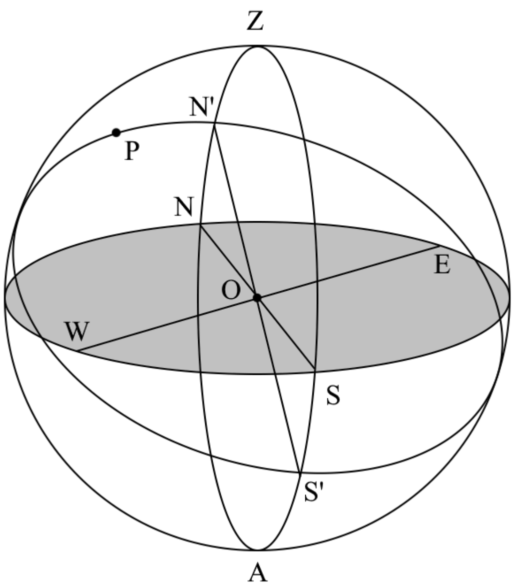
If we add l to t, we can see we get local solar day 175, matching with the local observation date.
If we had gotten for this value local solar day 174, we would try again but with $\Theta_{t=175.00}$ to be 1 higher. (In our case, $\Theta_{t=175.00} = 176$ sdy .)

Chapter 6

Horizontal Coordinates

Let's define a set of coordinates that we can use to describe the location of objects from the view of an observer on the ground.

6.1 The Hour Angle



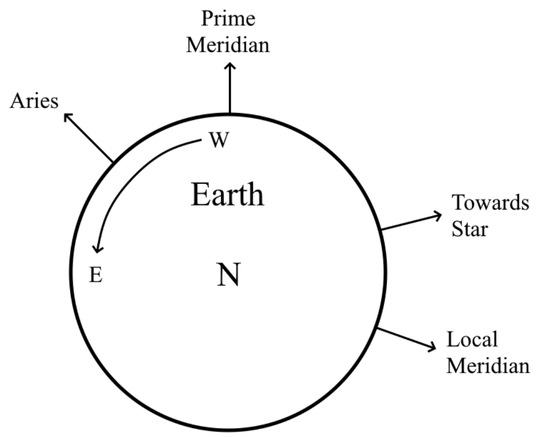
Consider this diagram.

This diagram depicts the celestial sphere around an observer at point O. The gray plane is the local horizon. ON, OS, OE, and OW, are the local directions of North, South, East, and West respectively. N' and S' depict the celestial poles, and thus the line $N'S'$ is the Earth's rotational axis, and therefore the plane perpendicular to N' and S' is the Earth's equatorial plane. Therefore, the angle $N'ON$ defines the observer's local latitude ϕ. Z and A are the observer's Zenith and Nadir respectively.

The plane (or circle) $NZSA$ defines the local meridian.

In the last chapter, we saw how we could define the sidereal time as the Western distance between Aries and the Meridian.

Now we are interested in the point P, which may be the location of a star or a planet. We can do the same thing, and we can define the angle between the Meridian and P, putting West as positive again. This angle is called the **hour angle** and is more rigorously defined as the angle between the meridian circle and the circle that goes through P, N', and S', which is called the *hour circle* of P. This is useful as it gives us a way to describe the location of P, specifically how East or West it is, in comparison to the meridian.



In this diagram, the Western angle between the Prime Meridian and Aries (the right ascension of the Prime Meridian) is the standard sidereal time (Θ), and the Western angle between the Local Meridian and Aries (the right ascension of the Local Meridian) is the local standard time (Θ_L), as we saw in the last chapter. The Eastern angle between Aries and the Star is, by definition, the right ascension of the star (α). Furthermore, the Eastern angle between the Prime Meridian and the Local Meridian is the longitude l. (Eastern angle meaning the angle measured in the Eastern direction.)

The Western angle between the Prime Meridian and the Star is then known as the Standard Hour Angle of the star (h), and the Western angle between the Local Meridian and the Star is known as the Local Hour Angle of the star (h_L).

It is therefore evident that

$$h_L = \Theta_L - \alpha \tag{6.1}$$

and because $\Theta_L = \Theta + l$ (equations 5.2 and 5.3):

$$\left.\begin{aligned} h_L &= \Theta + l - \alpha \\ &= h + l \end{aligned}\right\} \quad (6.2)$$

When $h_L = 0$, the star is coincident with the meridian, and the star is at the highest point in the sky. If $h_L = 180°$, the star is coincident with the lower meridian, and it is at the lowest point in the sky. If the star in question is the Sun, then the times at which $h_L = 0$ and $h_L = 180°$ are called *apparent noon* and *apparent midnight* respectively. These are not the same as the *mean noon* and *mean midnight*, the mean values are simply the average of the apparent values over the year. (Yes, this means noon and midnight aren't always at 12 : 00 and 00 : 00!)

Example 6.1 On planet P, on solar day 175 at local mean solar time 11 : 00 : 00, what was the local hour angle of star S, with right ascension 5^h?

Solution
175 dy 11 : 00 : 00 is 175.458333 dy.
We need the sidereal time, so by equation 5.4:

$$\begin{aligned} \Theta_L &= \frac{289.42 + 1}{289.42} \cdot 175.458333 - 0.5 \\ &= 175 \text{ sdy } 203° \ 14' \ 48.61'' \end{aligned}$$

Then, by equation 6.1:

$$\begin{aligned} h_L &= 203° \ 14' \ 48.61'' - 5^h \\ &= 128° \ 14' \ 48.61'' \end{aligned}$$

Example 6.2 On planet P, at some time during solar day 175, the hour angle of star S (with right acension 5^h) was $128° \, 14' \, 48.61''$. What was the local mean solar time?

Solution
By equation 6.1:

$$\begin{aligned}\Theta_L &= 128° \, 14' \, 48.61'' + 5^h \\ &= 203° \, 14' \, 48.61''\end{aligned}$$

We now follow example 5.3: we try $\Theta = 175 \text{ sdy } 203° \; 14' \; 48.61''$:

$$\begin{aligned}t &= (175 \text{ sdy } + 203° \, 14' \, 48.61'' + 180°) \cdot \frac{289.42}{290.42} \\ &= 175.458333 \text{ dy} \\ &= 175 \text{ dy } 11:00:00\end{aligned}$$

Which agrees with example 6.1.

Example 6.3 On planet P at standard time $t = 175.00$ dy , the Sun's Ecliptic Longitude λ_{Sun} was $217° \; 40' \; 36.24''$. What was the standard mean solar time of apparent noon on solar day 175 at $l = 0°E$? (The axial tilt ε of P is $25.5°$)

Solution
We mostly follow the previous example.
Because $l = 0°$, $h_L = h$.

$$h = \Theta - \alpha$$

Apparent noon is when h_L of the Sun $= 0°$, therefore at apparent noon, $\Theta = \alpha$. Using equations 1.1, 1.2, and 1.3, and setting $\beta = 0°$ from the definition of the Ecliptic, we find:

$$\Theta = \alpha = 214° \, 52' \, 37.04''.$$

Then, using the method of example 5.3, we try $\Theta = 175 \text{ sdy } 215° \; 25' \; 50.5''$.

$$\begin{aligned}t &= (175 \text{ sdy } 214° \, 52' \, 37.04'' + 180°) \cdot \frac{289.42}{290.42} \\ &= 175.4905 \text{ dy} \\ &= 175 \text{ dy } 11:46:22\end{aligned}$$

However, in this example, $t = 175 \text{ dy } 11:46:22 \neq 175.00$ dy ! Thus, our λ_{Sun} value would be off by some amount because the Sun would have moved during the $11h \; 46m \; 22s$. Thus, this time only works as a preliminary approximation, and we will have to repeat our calculations if we want a better result.

Example 6.3-II Using the fact that at $t = 175 \text{ dy } 11:46:22$, λ_{Sun} was $218° \; 17' \; 12.78''$, improve the approximation of the time of apparent noon.

Solution
Again, using equations 1.1, 1.2, and 1.3, we find:

$$\Theta = \alpha = 215° \, 28' \, 9.28''.$$

Then repeating the method of example 5.3,

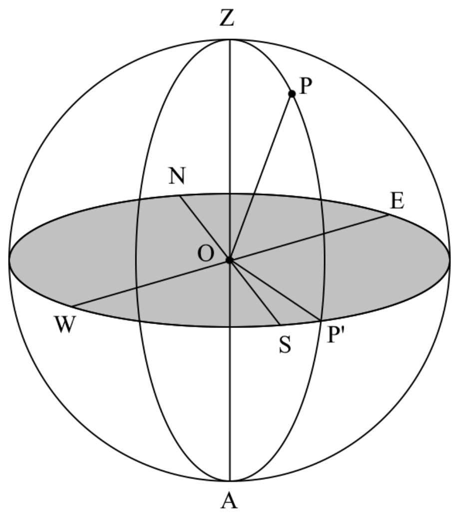
$$\begin{aligned} t &= (175 \ \ \text{sdy} \ \ 215^\circ \, 28' \, 9.28'' + 180^\circ) \cdot \frac{289.42}{290.42} \\ &= 175 \ \ \text{dy} \ \ 11:48:43 \end{aligned}$$

More repetition will improve our estimations even further, but they rapidly converge. (A third estimate using the longitude of the Sun at 11 : 48 : 43 gives the time 11 : 48 : 44.)

We can see that the time of apparent noon is not 12 : 00. This difference is called the *Equation of Time*, and the amount by it differs by changes throughout the year.

6.2 Horizontal Coordinates

Let's now get to defining our coordinates.



Let us say we want to describe the location of P. This can be done very simply with two angles, using spherical coordinates. First we drop P down onto the local horizon to form P'. Then we can describe the position of P by the angles NOP' and POP'.

The angle NOP' is called the *azimuth* (denoted A, not to be confused with the A in the diagram) and is measured from North with East as positive. The angle POP' is called the *altitude* (denoted a). This system of coordinates is called the **Horizonal Coordinate System**.

The way of calculating the azimuth and altitude is again given with rotation matrices, just like in chapter 1:

- **Equatorial to Horizontal:**

$$\begin{bmatrix} x_{\text{horizontal}} \\ y_{\text{horizontal}} \\ z_{\text{horizontal}} \end{bmatrix} = \begin{bmatrix} -\sin(\phi) & 0 & \cos(\phi) \\ 0 & -1 & 0 \\ \cos(\phi) & 0 & \sin(\phi) \end{bmatrix} \begin{bmatrix} \cos(\delta)\cos(h) \\ \cos(\delta)\sin(h) \\ \sin(\delta) \end{bmatrix} \tag{6.3}$$

Where ϕ is the latitude of observer and h is the **local** hour angle of P.

- **Horizontal to Equatorial (Does not change):**

$$\begin{bmatrix} \cos(\delta)\cos(h) \\ \cos(\delta)\sin(h) \\ \sin(\delta) \end{bmatrix} = \begin{bmatrix} -\sin(\phi) & 0 & \cos(\phi) \\ 0 & -1 & 0 \\ \cos(\phi) & 0 & \sin(\phi) \end{bmatrix} \begin{bmatrix} x_{\text{horizontal}} \\ y_{\text{horizontal}} \\ z_{\text{horizontal}} \end{bmatrix} \tag{6.4}$$

The hour angle is used because it factors in both the rotation of the Earth and the right ascension of the star (equation 6.1).

Note that in older textbooks, the azimuth is defined starting from South and going West. This is not the IAU's definition of the azimuth anymore, and therefore the azimuth values given in older books would have to be shifted by 180° to match the modern definition.

Example 6.4 On planet P at latitude 50°, on solar day 175 at local mean solar time $11:00:00$, what were the horizontal coordinates of star S, with right acension 5^h and declension $+30°$?

Solution
In example 6.1, we found the hour angle of S:

$$h = 128° \; 14' \; 48.61''$$

Thus, using equation 6.3:

$$\begin{bmatrix} x_{\text{horizontal}} \\ y_{\text{horizontal}} \\ z_{\text{horizontal}} \end{bmatrix} = \begin{bmatrix} -\sin(50°) & 0 & \cos(50°) \\ 0 & -1 & 0 \\ \cos(50°) & 0 & \sin(50°) \end{bmatrix} \begin{bmatrix} \cos(30°)\cos(128° \; 14' \; 48.61'') \\ \cos(30°)\sin(128° \; 14' \; 48.61'') \\ \sin(30°) \end{bmatrix}$$

Thus

$$\begin{bmatrix} x_{\text{horizontal}} \\ y_{\text{horizontal}} \\ z_{\text{horizontal}} \end{bmatrix} = \begin{bmatrix} 0.732080607 \\ -0.680134006 \\ 0.038415077 \end{bmatrix}$$

Now, using equation 1.2, we can find A and a:

$$\begin{aligned} \rho &= 1 \text{ (Celestial Sphere has arbitrary radius)} \\ A &= \arctan(-0.680134006, +0.732080607) \\ &= 317° \; 6' \; 23.78'' \\ a &= \arcsin(0.038415077/1) \\ &= 2° \; 12' \; 5.63'' \end{aligned}$$

We can see that S was in the Northwest ($270° < A < 360°$) and very close to the horizon.

6.3 The Sunrise Equation

Let's calculate the time of sunrise. "Sunrise" means that the Sun is at the horizon, or more specifically, the Eastern horizon. This means that the altitude a of the Sun is 0°, and the hour angle of the sun h is a negative number. (Remember, the hour angle is measured such that *West* is positive.)

Recall equation 6.3.

$$\begin{bmatrix} x_{\text{horizontal}} \\ y_{\text{horizontal}} \\ z_{\text{horizontal}} \end{bmatrix} = \begin{bmatrix} -\sin(\phi) & 0 & \cos(\phi) \\ 0 & -1 & 0 \\ \cos(\phi) & 0 & \sin(\phi) \end{bmatrix} \begin{bmatrix} \cos(\delta)\cos(h) \\ \cos(\delta)\sin(h) \\ \sin(\delta) \end{bmatrix}$$

Because $z_{\text{horizontal}} = \sin(a)$, when $a = 0°$, $z_{\text{horizontal}} = 0$.
So, let's set $z_{\text{horizontal}} = 0$ and solve for h.
Carrying out the matrix multiplication only for z:

$$\begin{aligned} z_{\text{horizontal}} = 0 &= \cos(\phi)\cos(\delta)\cos(h) + 0 \cdot \cos(\delta)\sin(h) + \sin(\phi)\sin(\delta) \\ &= \cos(\phi)\cos(\delta)\cos(h) + \sin(\phi)\sin(\delta) \\ \therefore \cos(h) &= -\frac{\sin(\phi)\sin(\delta)}{\cos(\phi)\cos(\delta)} \\ &= -\tan(\phi)\tan(\delta) \end{aligned} \tag{6.5}$$

Equation 6.5 is called the *Sunrise Equation.* However, it can be used to calculate the rising time of any celestial object, not just the Sun. If the positive arccosine value is taken, then the formula will calculate the setting time instead.

Notice that if $\phi > \arctan(\cot(\delta))$, then the sunrise equation predicts $\cos(h) > 1$ which is impossible. Likewise, if $\phi < -\arctan(\cot(\delta))$, then $\cos(h) < -1$ which is also impossible. This means that the object will not rise or set in these latitudes. In particular, if $\cos(h) > 1$, then the object is perpetually below the horizon. If $\cos(h) < -1$, then the object is perpetually up.

Example 6.5 On planet P at standard time $t = 175.00$ dy , the Sun's Ecliptic Longitude λ_{Sun} was $217° \, 40' \, 36.24''$. What was the mean solar time of sunrise on solar day 175 at $\phi = 50°N$ and $l = 0°E$?

Solution
To use the sunrise equation (equation 6.5), we clearly need δ, so, using equations 1.1, 1.2, and 1.3, we calculate the equatorial coordinates.

$$\delta = -15° \, 15' \, 21.30''$$
$$\alpha = 214\,° \, 52' \, 37.04''$$

Substituting ϕ and δ into equation 6.5:

$$\cos(h) = -\tan(50°)\tan(-15° \, 15' \, 21.3'')$$
$$\therefore h = \arccos(0.3250415)$$

Taking the negative arccosine value for h (negative because sunrise), we get:

$$h = -71° \, 1' \, 54.87''$$

From here we just need to find the mean time from the hour angle, so we follow example 6.2.

$$h_L = \Theta_L - \alpha$$

$$\begin{aligned} \therefore \Theta = \Theta_L &= h_L + \alpha = h + \alpha \\ &= -71° \, 1' \, 54.87'' + 214° \, 52' \, 37.04'' \\ &= 143° \, 50' \, 42.17'' \end{aligned}$$

$$\begin{aligned} t &= (175 \;\; \text{sdy} \;\; 143° \, 50' \, 42.17'' + 180°) \cdot \frac{289.42}{290.42} \\ &= 175 \;\; \text{dy} \;\; 07:03:13 \end{aligned}$$

Just as with example 6.3, this is just a preliminary approximation, and these calculations must be repeated for a more accurate time of sunrise.

At $t = 175$ dy $07:03:13$, $\lambda_{\text{Sun}} = 218° \, 2' \, 32.28''$.
Converting to equatorial coordinates:

$$\delta = -15° \, 23' \, 05.06''$$
$$\alpha = 215\,° \, 13' \, 54.03''$$

Thus the sunrise equation gives (again, taking the negative arccosine):

$$h = -70° \, 51' \, 26.21''$$

Now we follow example 6.2.

$$\Theta = 144^\circ\ 22'\ 27.82''$$
$$\therefore t = 175 \text{ dy } 07:05:19$$

Further repetition will better our approximations.

Note that the sunrise equation calculates when the center of the Sun (or any other object) is at the horizon. Because things have an apparent angular size in the sky, this means that the sunrise equation calculates at what hour angle exactly half of the object is visible. To calculate the precise time of first or last visibility, this angular size must be taken into account by calculating when the object's altitude would be 1 apparent radius (see chapter 4) below the horizon (instead of the altitude being precisely 0°), which we did not do here.

The equation can also be written to solve for the hour angle of an object at any altitude a:

$$\cos(h) = \frac{\sin(a) - \sin(\phi)\sin(\delta)}{\cos(\phi)\cos(\delta)} \tag{6.5*}$$

Furthermore, objects near the horizon have their positions significantly altered by atmospheric refraction (upto about $30'$ for our Earth), which depends on the density and composition of the atmosphere and the specific weather conditions of the time and location. However, this is far too complicated to go into any detail here, especially since existing formulae only apply to the Earth's atmosphere and in a worldbuilding setting they won't be accurate, so we will be assuming an airless environment in our calculations.

6.4 The Terminator

The *terminator* is the line separating night and day at any moment in time. Let's say at standard solar time t, the standard sidereal time was Θ. To find the terminator, we find the places where the Sun is rising or setting, i.e. we use the sunrise equation:

$$\cos(h) = -\tan(\phi)\tan(\delta)$$

Therefore:

$$h = \begin{cases} -\arccos(-\tan(\phi)\tan(\delta)) & \text{Sunrise} \\ \arccos(-\tan(\phi)\tan(\delta)) & \text{Sunset} \end{cases}$$

but by equation 6.2, $h = \Theta + l - \alpha$, so:

$$l = \begin{cases} -\arccos(-\tan(\phi)\tan(\delta)) - \Theta + \alpha & \text{Sunrise} \\ \arccos(-\tan(\phi)\tan(\delta)) - \Theta + \alpha & \text{Sunset} \end{cases} \tag{6.6}$$

Example 6.6 Find the terminator of the Earth on June 21, 2023 at standard solar time 14 : 56 : 00. $\lambda_{\text{Sun}} = 90^\circ$, $\Theta = 133^\circ\ 33'\ 1''$, and $\varepsilon = 23.44^\circ$.

Solution
By equation 1.3:

$$\alpha = 90^\circ$$
$$\delta = 23^\circ\ 26'\ 24''$$

Thus the limits of sunrise and sundown are:

$$\phi = \pm\arctan(\cot(23^\circ\ 26'\ 24'')) = \pm 66^\circ\ 33'\ 36''$$

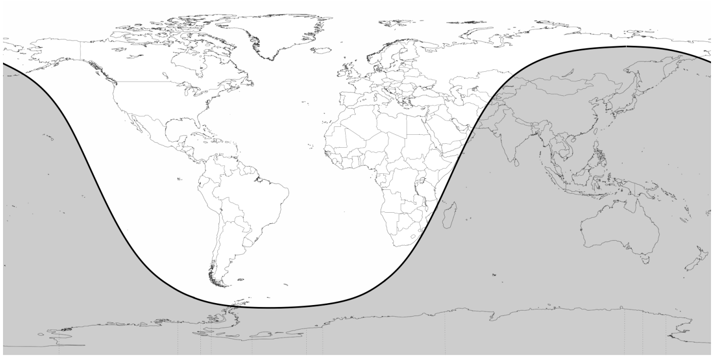
Now we use equation 6.6. For $\phi = 0°$:

$$l = \begin{cases} -\arccos(-\tan(0°)\tan(23°\ 26'\ 24'')) - 133°\ 33'\ 1'' + 90° & \text{Sunrise} \\ = 226°\ 26'\ 59'' & \text{Sunrise} \\ \arccos(-\tan(0°)\tan(23°\ 26'\ 24'')) - 133°\ 33'\ 1'' + 90° & \text{Sunset} \\ = 46°\ 26'\ 59'' & \text{Sunset} \end{cases}$$

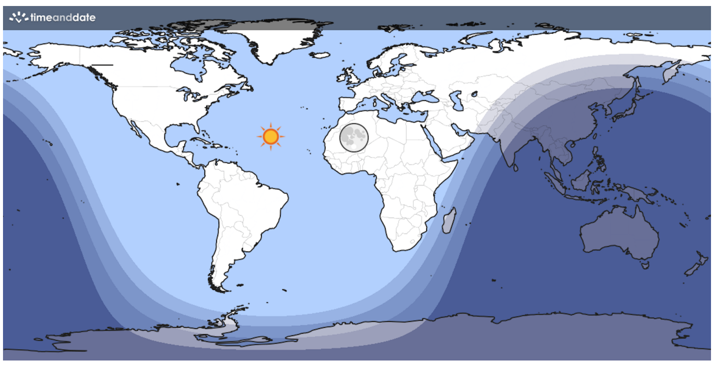
Filling out for $66°\ 33'\ 36'' \leq \phi \leq 66°\ 33'\ 36''$:

ϕ	l of Sunrise	l of Sundown
$66°\ 33'\ 36''$	$136°\ 27'$	$136°\ 27'$
$60°$	$177°\ 47'$	$95°\ 7'$
$50°$	$195°\ 20'$	$77°\ 34'$
$40°$	$205°\ 7'$	$67°\ 47'$
$30°$	$211°\ 57'$	$60°\ 57'$
$20°$	$217°\ 22'$	$55°\ 32'$
$10°$	$222°\ 4'$	$50°\ 50'$
$0°$	$226°\ 27'$	$46°\ 27'$
$-10°$	$230°\ 50'$	$42°\ 4'$
$-20°$	$235°\ 32'$	$37°\ 22'$
$-30°$	$240°\ 57'$	$31°\ 57'$
$-40°$	$247°\ 47'$	$25°\ 7'$
$-50°$	$257°\ 34'$	$15°\ 20'$
$-60°$	$275°\ 7'$	$357°\ 47'$
$-66°\ 33'\ 36''$	$316°\ 27'$	$316°\ 27'$

Plotting all these points on a map and connecting the dots, we get:



The Earth's terminator on June 21, 2023 at 14 : 56 : 00.
It is daytime in the light part and nighttime in the dark part.



The Earth's terminator on June 21, 2023 at 14 : 56 : 00 from *timeanddate.com*.
On this map, our calculations correspond to the border of the lightest shade of black.
The Sun and Moon icons show the subsolar and sublunar points (to be discussed later).

6.5 The Ascending Sign

The part of the ecliptic that is rising at any given time and location is called the *ascendant* (denoted *Asc.*) at that time and location. This is often simplified to the rising Zodiac sign (the *ascending sign*) and holds great importance in real world astrology.

In order for something to be rising, it must satisfy the sunrise equation:

$$\cos(h) = -\tan(\phi)\tan(\delta)$$

But by equation 6.1:

$$\begin{aligned}
\cos(\Theta_L - \alpha) &= -\tan(\phi)\tan(\delta) \\
\cos(\Theta_L)\cos(\alpha) + \sin(\Theta_L)\sin(\alpha) &= -\tan(\phi)\tan(\delta) \\
-\cos(\Theta_L) - \sin(\Theta_L)\tan(\alpha) &= \frac{\tan(\phi)\sin(\delta)}{\cos(\alpha)\cos(\delta)} \qquad (6.7)
\end{aligned}$$

Also, by equation 1.3, a point on the ecliptic with longitude λ ($\beta = 0°$) has equatorial coordinates:

$$\begin{aligned}
\alpha &= \arctan(\cos(\varepsilon)\sin(\lambda), \cos(\lambda)) \\
&= \arctan(\cos(\varepsilon)\tan(\lambda)) \quad \text{(Introduces ambiguities)} \\
\delta &= \arcsin(\sin(\varepsilon)\sin(\lambda))
\end{aligned}$$

Substituting these into equation 6.7:

$$-\cos(\Theta_L) - \sin(\Theta_L)\cos(\varepsilon)\tan(\lambda) = \frac{\tan(\phi)\sin(\varepsilon)\sin(\lambda)}{\cos(\alpha)\cos(\delta)}$$

But, $\cos(\alpha)\cos(\delta)$ is just $x_{\text{equatorial}}$ (by equation 1.1) which is just equal to x_{ecliptic} (by equation 1.4) which is $\cos(\lambda)\cos(\beta)$ (by equation 1.4) which becomes $\cos(\lambda)$ (because $\beta = 0°$). Thus:

$$\begin{aligned}
-\cos(\Theta_L) - \sin(\Theta_L)\cos(\varepsilon)\tan(\lambda) &= \frac{\tan(\phi)\sin(\varepsilon)\sin(\lambda)}{\cos(\lambda)} \\
&= \tan(\phi)\sin(\varepsilon)\tan(\lambda) \\
\therefore -\frac{\cos(\Theta_L)}{\tan(\lambda)} - \sin(\Theta_L)\cos(\varepsilon) &= \tan(\phi)\sin(\varepsilon) \\
\therefore \tan(\lambda) &= \frac{-\cos(\Theta_L)}{\tan(\phi)\sin(\varepsilon) + \sin(\Theta_L)\cos(\varepsilon)}
\end{aligned}$$

Splitting $\tan()$ into $\sin()/\cos()$ and taking the two argument arctangent, we get:

$$\lambda = \arctan\left(-\cos(\Theta_L), \tan(\phi)\sin(\varepsilon) + \sin(\Theta_L)\cos(\varepsilon)\right) \qquad (6.8)$$

To ensure a negative value for h (since the ecliptic must be rising), we employ one final correction:

$$\lambda_{Asc} = \begin{cases} \lambda + 180° & \lambda < 180° \\ \lambda - 180° & \lambda > 180° \end{cases} \qquad (6.9)$$

Example 6.7 Find the ascending sign at standard solar time $11:00$ on January 1, 2024 at Dubai, UAE ($\phi = +25°, l = 55°$). $\Theta = 265°\ 36'\ 15''$ and $\varepsilon = 23.44°$.

Solution
We need the local sidereal time, so by equation 5.3:

$$\Theta_L = 265°\ 36'\ 15'' + 55° = 320°\ 36'\ 15''$$

Then, by equation 6.8:

$$\begin{aligned}\lambda &= \arctan\left(-\cos(320°\ 36'\ 15''), \tan(25°)\sin(23.44°) + \sin(320°\ 36'\ 15'')\cos(23.44°)\right)\\ &= 242°\ 49'\ 13''\end{aligned}$$

Then, by equation 6.9:

$$\begin{aligned}\lambda_{Asc} &= 242°\ 49'\ 13'' - 180°\\ &= 62°\ 49'\ 13''\end{aligned}$$

Which falls in the sign Gemini ($60° < \lambda < 90°$), which was indeed the rising sign in Dubai at January 1, 2024 $15:00$ ($\text{UTC} + 4:00$).

6.6 Heliacal Rising

Because the Earth orbits the Sun, different parts of the sky are completely blocked out by the glare of the Sun in different seasons. The *heliacal rising* of a star happens when a star becomes visible for the first time again after being blocked out by the glare of the Sun for some time.

Because the Sun's ecliptic longitude increases (i.e. the Sun moves East), the Stars with fixed right ascensions and declinations to the East of the Sun will eventually be blocked by the Sun's light, and re-emerge to the West of the Sun when it is visible again. Because a Western elongation implies a morning star, the heliacal rising happens when the Star is visible in the dawn again. The star will then rise earlier and earlier before the Sun, until it loops all the way around and becomes lost in the glare of the Sun again.

Let's try and find out when the heliacal rising of a star with right ascension α and declination δ will happen at a location with latitude ϕ.

We estimate the heliacal rising to be the point when the star would rise exactly at sunrise, but in reality it is some days after this time because the Star still needs some time to be visible. Let's first find out when the star would rise. The rising time of a point with right ascension α and declination δ is given by the sunrise equation and equation 6.1:

$$\Theta_{L\ \text{Star-rise}} = -\arccos(-\tan(\phi)\tan(\delta)) + \alpha \tag{6.10}$$

Then we just have to find the λ for which the Sun would rise at exactly $\Theta_{\text{Star-rise}}$, because then the date can be found by the method of example 2.3. But, this problem has already been solved by equation 6.8 and 6.9! Therefore to get the date of heliacal rising, we simply find what the ascending sign is at that location for the time of starrise, and calculate the date for which the Sun is at that longitude.

Example 6.8 Calculate the date of the heliacal rising of the star Sirius ($\alpha = 06^h\ 45^m\ 09^s$, $\delta = -16°\ 42'\ 58''$) in Egypt ($\phi = 30°$).
Use $\varepsilon = 23.44°$, $e = 0.0167$, and $T = 365.24$ dy for the Earth. The time of the last periapsis of Earth was January 3, 2024.

Solution
By equation 6.10:

$$\begin{aligned}\Theta_{L \text{ Star-rise}} &= -\arccos(-\tan(30°)\tan(-16°\ 42'\ 58'')) + 06^h\ 45^m\ 09^s \\ &= 21°\ 16'\ 21''\end{aligned}$$

Then, by equation 6.8:

$$\begin{aligned}\lambda &= \arctan\left(-\cos(21°\ 16'\ 21''), \tan(30°)\sin(23.44°) + \sin(21°\ 16'\ 21'')\cos(23.44°)\right) \\ &= 301°\ 7'\ 3''\end{aligned}$$

Then, by equation 6.9:

$$\begin{aligned}\lambda_{Asc} &= 301°\ 7'\ 3'' - 180° \\ &= 121°\ 7'\ 3''\end{aligned}$$

Thus the longitude of the Sun should be 121° 7′ (3″ ignored just for brevity) if it were to rise at the same time as Sirius. If $\lambda_{\text{Sun}} = 121°\ 7'$, then $\lambda_{\text{Earth Heliocentric}} = 121°\ 7' + 180° = 301°\ 7'$ (Equation 1.5). Now we follow example 2.3.

$$\begin{aligned}\nu &= 301°\ 7' - 102°\ 56'\ 49.9'' \\ &= 198°\ 10'\ 13.29''\end{aligned}$$

ν is bigger than 180°, so we take the negative arccosine in equation 2.16:

$$\begin{aligned}E &= -\arccos\left(\frac{0.0167 + \cos(198°\ 10'\ 13.29'')}{1 + 0.0167\cos(198°\ 10'\ 13.29'')}\right) + 2\pi \\ &= 3.463975 \text{ rad} \\ \therefore M &= 3.463975 - 0.0167\sin(3.463975) \\ &= 3.469266 \text{ rad} \\ \therefore t &= \frac{3.469266}{2\pi} \cdot 365.24 \text{ dy} \\ &= 202 \text{ dy}\end{aligned}$$

202 dy after January 3, 2024 is July 23, 2024. Adding about 10 days to give Sirius time to actually be visible, we get August 2, 2024. This is almost exact: the heliacal rising of Sirius happens around August 1 in Egypt.

The ancient Egyptians thought of this event to have immense significance because it happened just before the beginning of the flooding season of the Nile river (around late Summer) by coincidence.

6.7 The Medium Coeli

Similar to the Ascending sign, the Zodiac sign that is highest in the sky is known as the *Medium Coeli* (or "Midheaven") (denoted $M.C.$).

If something is highest in the sky, it is passing the meridian and thus the hour angle is 0°. Thus, by equation 6.1:

$$\Theta_L = \alpha$$

But from earlier we know that for a point on the ecliptic with longitiude λ, α is given by:

$$\alpha = \arctan(\cos(\varepsilon)\sin(\lambda), \cos(\lambda))$$

And thus:

$$\tan(\Theta_L) = \cos(\varepsilon)\tan(\lambda)$$

Splitting tan() into sin()/ cos() and taking the two argument arctangent:

$$\lambda = \arctan\left(\frac{\sin(\Theta_L)}{\cos(\varepsilon)}, \cos(\Theta_L)\right) \tag{6.11}$$

Example 6.9 Find the Midheaven at standard solar time 11 : 00 on January 1, 2024 at Dubai, UAE ($\phi = +25°, l = 55°$). $\Theta = 265°\ 36'\ 15''$ and $\varepsilon = 23.44°$.

Solution
The local sidereal time was found in example 6.7 to be $320°\ 36'\ 15''$. Thus by equation 6.11:

$$\begin{aligned}\lambda_{M.C.} &= \arctan\left(\frac{\sin(320°\ 36'\ 15'')}{\cos(23.44°)}, \cos(320°\ 36'\ 15'')\right)\\ &= 318°\ 9'\ 59''\end{aligned}$$

Which falls in the sign Aquarius ($300° < \lambda < 330°$), which was indeed the Midheaven sign in Dubai at January 1, 2024 15 : 00 (UTC + 4 : 00).

6.8 The Subsolar Point

In some regions of the world, there are times when the Sun is directly overhead (i.e. at the Zenith) and objects leave no shadow on the ground. This phenomenon is called the *Lahaina noon*) and the point where this occurs on the Earth is called the *subsolar point*. Let's try to find the subsolar time at any time.

Say that at (solar) time t, the Sun was at longitude λ, which can be converted into right ascension α and declination δ. If the Sun is at the Zenith, then the x and y coordinates of the Sun in the horizontal frame would be 0 and only the z component would exist. From equation 6.3 we can deduce:

$$\begin{aligned}x_{\text{horizontal}} &= -\sin(\phi)\cos(\delta)\cos(h) + \cos(\phi)\sin(\delta) = 0\\ y_{\text{horizontal}} &= \cos(\delta)\sin(h) = 0\end{aligned}$$

If the Sun is at the Zenith then it is also at the meridian and thus it would be the apparent noon at that place. Thus, $h = 0°$. Substituting this gives:

$$\begin{aligned}x_{\text{horizontal}} &= -\sin(\phi)\cos(\delta) + \cos(\phi)\sin(\delta)\\ y_{\text{horizontal}} &= 0\end{aligned}$$

The y coordinate is already 0. Setting $x_{\text{horizontal}} = 0$ and solving for ϕ we get:

$$\tan(\phi) = \tan(\delta)$$
$$\therefore \phi_{\text{Subsolar}} = \delta \tag{6.12}$$

Because the declension of the Sun cannot exceed the axial tilt of the planet, Lahaina noon only occurs in regions with latitudes between $+\varepsilon$ and $-\varepsilon$ (the Tropics of Cancer and Capricorn).

Because $h = 0°$, $\Theta_L = \alpha$ (by equation 6.1). Thus, if we know the standard sidereal time Θ at time t, we can calculate the longitude by:

$$l_{\text{Subsolar}} = \alpha - \Theta \tag{6.13}$$

Which comes from equation 6.2.

Example 6.10 Find the subsolar point if $\lambda_{\text{Sun}} = 66° \ 40'$ and $\Theta = 24° \ 55' \ 27''$. Use $\varepsilon = 23.44°$.

Solution
By equation 1.3:

$$\alpha = 4^h \ 19^m \ 17^s$$
$$\delta = 21° \ 25' \ 25''$$

Thus by equation 6.12 and 6.13:

$$\phi = \delta = 21° \ 25' \ 25''$$
$$l = \Theta - \alpha = 39° \ 53' \ 42''$$

Which is in Mecca, Saudi Arabia.

Now to find the date when Lahaina noon occurs at a specific location. We have from earlier:

$$\delta = \phi$$

But also, by equation 1.1 we know:

$$x_{\text{equatorial}} = \cos(\delta)\cos(\alpha)$$
$$y_{\text{equatorial}} = \cos(\delta)\sin(\alpha)$$
$$z_{\text{equatorial}} = \sin(\delta)$$

Therefore, using equation 1.4:

$$z_{\text{ecliptic}} = -\sin(\varepsilon)\cos(\delta)\sin(\alpha) + \cos(\varepsilon)\sin(\delta)$$

But we also know that since β of the Sun is $0°$, $z_{\text{ecliptic}} = \sin(0°) = 0$. Thus:

$$0 = -\sin(\varepsilon)\cos(\delta)\sin(\alpha) + \cos(\varepsilon)\sin(\delta)$$
$$\therefore \sin(\alpha) = \frac{\cos(\varepsilon)\sin(\delta)}{\sin(\varepsilon)\cos(\delta)}$$

So we take the arcsine. arcsin() has two solutions:

$$\alpha_1 = \arcsin\left(\frac{\cos(\varepsilon)\sin(\delta)}{\sin(\varepsilon)\cos(\delta)}\right) \tag{6.14}$$
$$\alpha_2 = 12^h - \alpha_1 \tag{6.15}$$

Therfore at any location, if Lahaina noon is possible, there will be two dates when this happens. (Unless one is *exactly* at the Tropic of Cancer or Capricorn. Then α_1 works out to be equal to α_2.)

From α and δ we can find λ and then find the date by means of example 2.3. However, notice that the coordinates of the Sun depend only on ϕ and not l. This means that this formula predicts that an entire line of latitude will all experience Lahaina noon, however this is impossible as there is only one subsolar point for a specific location of the Sun, not a whole latitude line. In reality, what's happening here is that on average the whole latitude line will experience Lahaina noon but the true subsolar point depends on the specific rotation of the Earth (i.e. the sidereal time). However, the change in declination of the Sun is so slow (it takes one full year to complete the cycle of declination) that the change of latitude of the subsolar point in the course of a sidereal day is almost negligible, and thus we can approximate that it is the whole latitude line that experiences Lahaina noon. This approximation works better the more sidereal days there are in a planet year.

Example 6.11 Find the λ values of the Sun for which there is Lahaina noon in Mecca, Saudi Arabia. ($\phi = 21^\circ\ 25'\ 25''$).

Solution
By equation 6.10:

$$\delta = \phi = 24^\circ\ 25'\ 25''$$

Then, by equations 6.14 and 6.15:

$$\begin{aligned}\alpha_1 &= \arcsin\left(\frac{\cos(23.44^\circ)\sin(24^\circ\ 25'\ 25'')}{\sin(23.44^\circ)\cos(24^\circ\ 25'\ 25'')}\right)\\ &= 4^h\ 19^m\ 17^s\\ \alpha_2 &= 12^h - 4^h\ 19^m\ 17^s\\ &= 7^h\ 40^m\ 43^s\end{aligned}$$

Then, by equation 1.4:

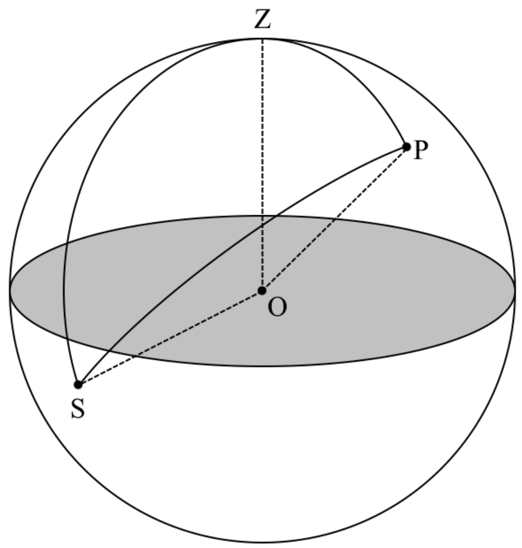
$$\begin{aligned}\lambda_1 &= 66^\circ\ 40'\\ \lambda_2 &= 113^\circ\ 20'\end{aligned}$$

λ_1 is the value of λ from the previous example.

These can be converted to dates by following example 2.3, and should result in dates around May 27, 2024 and July 15, 2024. At these dates, at the moment of local apparent noon at Mecca, the Sun is directly overhead Mecca, and therefore for anyone in any other place in the world where the Sun is up, their shadow would point perfectly away from Mecca. The direction of the Muslim prayer (the *Qibla*), which is in the direction of Mecca, can be determined then by the direction opposite to one's shadow on these dates.

6.9 Lighting Direction

Here is another thing we can do with horizontal coordinates: in chapter 4, we learned how to calculate how much of a planet was visible (the phase) and even which side of the planet was illuminated. Let us now figure out specifically which direction the light was coming from.



In this diagram, the Sun is at point S, and the planet of interest is at point P. Z is the zenith. The arc connecting the Sun and the Planet is shown. What we are interested in is in which direction is S from P? More specifically, we are interested in the angle ZPS.

This can be calculated with the spherical law of cosines:

$$\cos(ZPS) = \frac{\cos(ZOS) - \cos(ZOP)\cos(SOP)}{\sin(ZOP)\sin(SOP)}$$

The angle ZOS and ZOP are called the *zenith distances* (denoted ζ) of S and P and can be calculated by:

$$\zeta = 90^\circ - a \tag{6.16}$$

Therefore, the equation becomes:

$$ZPS = \arccos\left(\frac{\sin(a_S) - \sin(a_P)\cos(SOP)}{\cos(a_P)\sin(SOP)}\right) \tag{6.17}$$

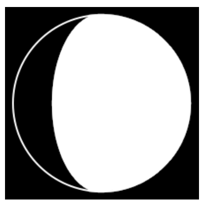
The angle SOP can be calculated by equation 4.3.

Example 6.12 Recall example 4.1 where we calculated the phase of Venus on January 1, 2024. Calculate how Venus would have looked from London, UK ($\phi = 51^\circ, l = 0^\circ$) at $06:00$ ($\Theta = 190^\circ\ 23'\ 56.5''$).

Solution
Recall from example 4.1:

$$\begin{aligned}
\alpha_V &= 16^h\ 07^m\ 26^s \\
\delta_V &= -18^\circ\ 57'\ 52'' \\
\alpha_S &= 18^h\ 46^m\ 38^s \\
\delta_S &= -23^\circ\ 0'\ 10'' \\
SOV &= 37^\circ\ 16'\ \ 7.9''
\end{aligned}$$

Where S is the Sun and V is Venus.
We found out the the phase of Venus was 78%:



Now, let's figure out the lighting direction. Using equation 6.1:

$$\begin{aligned}
h_V &= 190^\circ\ 23'\ 56.5'' - 16^h\ 07^m\ 26^s = -51^\circ\ 27'\ 33.5'' \\
h_S &= 190^\circ\ 23'\ 56.5'' - 18^h\ 46^m\ 38^s = -91^\circ\ 15'\ 33.5''
\end{aligned}$$

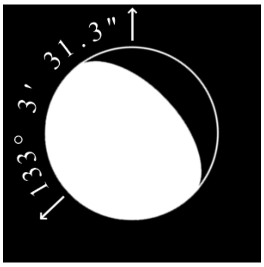
Now, using equation 6.3 (See example 6.4 for more detail):

$$\begin{aligned} A_V &= 131^\circ\, 50'\, 46.3'' \\ a_V &= 6^\circ\, 47'\, 32.0'' \\ A_S &= 104^\circ\, 2'\, 39.1'' \\ a_S &= -18^\circ\, 26'\, 47.8'' \end{aligned}$$

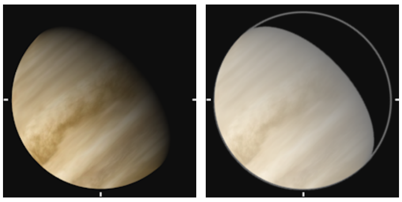
And then finally by equation 6.17:

$$\begin{aligned} ZPS &= \arccos\left(\frac{\sin(-18^\circ\, 26'\, 47.8'') - \sin(6^\circ\, 47'\, 32.0'')\cos(37^\circ\, 16'\ 7.9'')}{\cos(6^\circ\, 47'\, 32.0'')\sin(37^\circ\, 16'\ 7.9'')}\right) \\ &= 133^\circ\, 3'\, 31.3'' \end{aligned}$$

Thus, the light was shining from an angle of $133^\circ\, 3'\, 31.3''$ from the Zenith. Keeping in mind that since the Sun's azimuth is less than that of Venus, the Sun must be shining from the left:

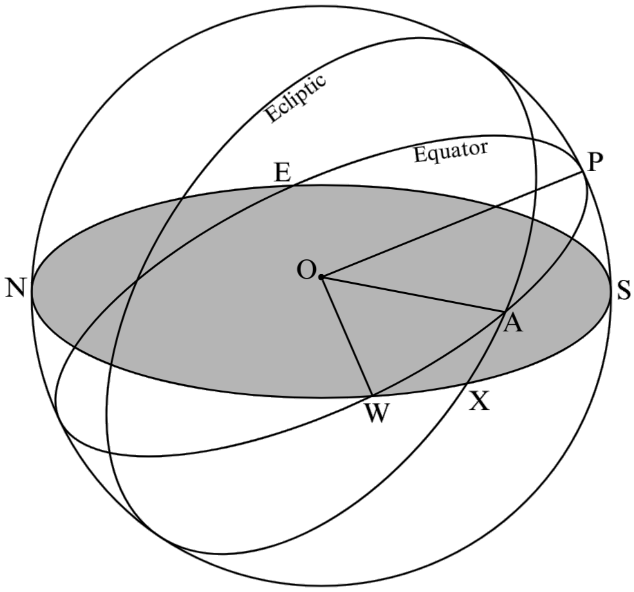


Comparing with *Stellarium* (a planetarium software):



Stellarium image on the left, Stellarium image overlaid with our prediction on the right.

6.10 Tilt of the Ecliptic to the Horizon



In this diagram, O is the position of the observer, the great circle $NESW$ are the horizon, the great circle NPS is the local meridian, the great circle AX is the Ecliptic, and the great circle EPW is the Celestial Equator. Therefore angle WAX is the axial tilt of the Earth, ε. Additionally, N and S are the points where the meridian meets the horizon, and therefore are North and South. E and W are the points where the equator meets the horizon, and therefore are East and West. Also, A is the point of spring equinox, P is where the meridian meets the equator, and X is the point where the ecliptic meets the horizon. This means that angle $AWX = 90° - \phi$, where ϕ is the local latitude.

Let us try and find the angle of the tilt of the ecliptic to the horizon, AXS, which we will denote I.

Notice in the spherical triangle AXW, we can apply the spherical law of cosines and obtain:

$$\cos(AXW) = -\cos(AWX)\cos(WAX) + \sin(AWX)\sin(WAX)\cos(WOA)$$

Evidently, $AXW = 180° - I$ and, as previously discussed, $AWX = 90° - \phi$ and $WAX = \varepsilon$. Therefore, the above equation becomes:

$$\cos(I) = \sin(\phi)\cos(\varepsilon) - \cos(\phi)\sin(\varepsilon)\cos(WOA)$$

Now let's figure out angle WOA. Since A is the point of vernal equinox, angle AOP is the right ascension of the meridian, or as discussed in chapter 5, the local sidereal time Θ_L. Also, because angle WOP is the angle from West to South, it is evidently 90°. Therefore $WOA = 90° - \Theta_L$. Now we can substitute this value into the equation above and obtain:

$$\cos(I) = \sin(\phi)\cos(\varepsilon) - \cos(\phi)\sin(\varepsilon)\sin(\Theta_L) \tag{6.18}$$

Example 6.13 Find the tilt of the ecliptic to the horizon at January 1, 2024 at 04 : 00 in London, UK ($\phi = 51°, l = 0°$). Use $\varepsilon = 23.44°$

Solution
The sidereal time at this time was:

$$\Theta = 25°\,55'\,45''$$

Since $l = 0°$, $\Theta_L = \Theta$. Therefore by equation 6.18:

$$\begin{aligned}\cos(I) &= \sin(51°)\cos(23.44°) - \cos(51°)\sin(23.44°)\sin(25°\,55'\,45'') \\ &= 0.603552 \\ \therefore I &= 52°\,52'\,31''\end{aligned}$$

Planets that are close to the Sun (that have low elongations) will be easier to observe when the tilt of the ecliptic is high at sunrise or sunset as that means the altitude of the planet will be higher at sunset or sunrise compared to when the ecliptic has a low tilt.

Example 6.14 Determine if evening planets close to the Sun are better visible in Spring sunsets or in Autumn sunsets in the Northern hemisphere.

Solution
In equation 6.18, or I to be high, the value of

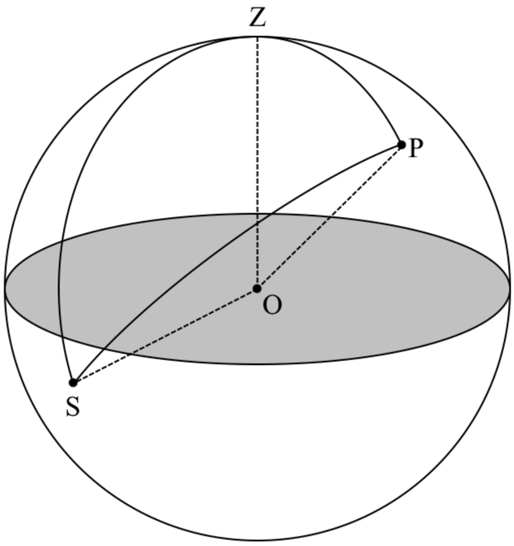
$$\sin(\phi)\cos(\varepsilon) - \cos(\phi)\sin(\varepsilon)\sin(\Theta_L)$$

must be low. Since ϕ is a fixed positive value (positive because Northern hemisphere), and ε is also fixed, the only variable is Θ_L. Therefore, to minimize the above value, $\cos(\phi)\sin(\varepsilon)\sin(\Theta_L)$ must be maximum, and therefore $\sin(\Theta_L)$ must be maximum. This happens when $\Theta_L = 90°$.

$\Theta_L = 90°$ at sunset means that the right ascension of the meridian is 90° at sunset, and that means the meridian (i.e. the highest point of the equator in the sky, labeled point P in the above diagram) is 90° East of the Spring Equinox point.

Since the meridian is located in the South in the Northern hemisphere, that means that the Spring Equinox point must be in the West, i.e. in the direction where the Sun sets. Therefore the Sun must be near the Spring Equinox point, and therefore the tilt of the ecliptic is higher in Spring sunsets than Autumn sunsets in the Northern hemisphere.

6.11 Tilt of the Equatorial Grid to the Horizontal Grid



In this diagram, point S is the point of interest, P is the celestial North Pole, and Z is the zenith. The tilt of the equatorial grid to the horizontal grid at point S can be given as the angle PSZ, which can be calculated with the spherical law of cosines as:

$$\cos(PSZ) = \frac{\cos(ZOP) - \cos(ZOS)\cos(SOP)}{\sin(ZOS)\sin(SOP)}$$

Where, ZOP is $90° - \phi$ where ϕ is the local latitude, SOP is $90° - \delta$ where δ is the declination of S, and as discussed in section 6.9, ZOS is the zenith distance ζ of the point S.

Then, because $\zeta = 90° - a$ (equation 6.16), the above equation becomes:

$$\cos(PSZ) = \frac{\sin(\phi) - \sin(a)\sin(\delta)}{\cos(a)\cos(\delta)}$$

Thus:

$$PSZ = \arccos\left(\frac{\sin(\phi) - \sin(a)\sin(\delta)}{\cos(a)\cos(\delta)}\right) \tag{6.19}$$

Example 6.15 Find the tilt of the equatorial grid at the location of Venus on January 1, 2024 from London, UK ($\phi = 51°, l = 0°$) at 06 : 00.

Solution
From example 6.12, we have:

$$\delta_V = -18° \, 57' \, 52''$$
$$a_V = 6° \, 47' \, 32.0''$$

Thus, by equation 6.19:

$$\begin{aligned} ZPS &= \arccos\left(\frac{\sin(51°) - \sin(6° \, 47' \, 32.0'')\sin(-18° \, 57' \, 52'')}{\cos(6° \, 47' \, 32.0'')\cos(-18° \, 57' \, 52'')}\right) \\ &= 30° \, 20' \, 55'' \end{aligned}$$

Because the azimuth of Venus was $A_V = 131° \, 50' \, 46.3''$, which is in the Southeast, the celestial north pole must be to the left of it (because the celestial north pole always points towards North), meaning the equatorial grid was tilted $30° \, 20' \, 55''$ to the left of the horizontal grid at that location at that time, or in other words, the horizontal grid was tilted $30° \, 20' \, 55''$ to the right of the equatorial grid.

Chapter 7

The Shape of the Earth

We will now refine our calculations for observations on the surface of the Earth, so we must know the figure of the Earth.

7.1 The Ellipsoid

As mentioned in chapter 3, the Earth is roughly a squished sphere, meaning it is an ellipsoid (in particular, a spheroid). Due to the Earth's rotation, its polar radius its smaller than its equatorial radius. The amount of flattening of the Earth, called the *flattening* (f), is calculated with the following formula:

$$f = \frac{a-b}{a} = 1 - \frac{b}{a} \tag{7.1}$$

Where $b =$ the polar radius, and $a =$ the equatorial radius.

The ellipsoid, being a solid of rotation where every vertical (meridional) cross section is an ellipse, can be simplified to be an ellipse for most things:
Since the eccentricity e of an ellipse is defined by

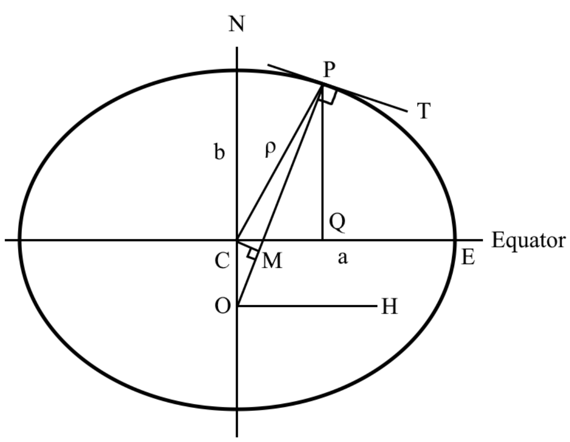
$$e^2 = 1 - \frac{b^2}{a^2}$$

it follows that

$$e^2 = 1 - (1-f)^2$$
$$\therefore e = \sqrt{2f - f^2}. \tag{7.2}$$

7.2 Latitude

How do we define latitude on a spheroid?



In the diagram, where the Earth is depicted as an ellipse, we can see two ways to define the latitude of the point P. The *geocentric* latitude is given by the angle PCE, and the *geodetic* or *geographical* latitude given by the angle POH, where the line OP is perpendicular to the tangent at P, PT, and OH is parallel to CE.

When one refers to latitude, usually one is referring to the *geodetic* (*geographical*) latitude (ϕ). This definition is used as this definition of latitude makes sure that a perpendicular change in the observer's elevation does not change the observer's latitude. This is the value of latitude we used in chapter 6.

However, the geocentric latitude is more convenient in some situations. So, given a geodetic latitude, how do we find the geocentric latitude? Well, consider the equation of the ellipse with semi-major axis a and semi-minor axis b:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{7.3}$$

Because PO is perpendicular to the tangent PT, the slope of PO must be $-1/m_{PT}$. In other words:

$$\tan(\phi) = -\frac{dx}{dy}$$

And from the triangle PCQ we can deduce:

$$\tan(\phi') = \frac{y}{x}$$

Implicitly differentiating equation 7.3 we obtain:

$$\begin{aligned}
\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} &= 0 \\
\therefore \frac{2x}{a^2} &= -\frac{2y}{b^2}\frac{dy}{dx} \\
\therefore \frac{a^2}{x} &= -\frac{b^2}{y}\frac{dx}{dy} \\
\therefore \frac{y}{x} &= -\frac{b^2}{a^2}\frac{dx}{dy}
\end{aligned}$$

Therefore:

$$\tan(\phi') = \frac{b^2}{a^2} \cdot \tan(\phi) \tag{7.4}$$

It can be further proven with simple geometry that the difference between ϕ and ϕ', i.e. $\phi - \phi'$, called the *reduction in latitude*, is equal to the angle CPO in the diagram.

7.3 The Radius of the Earth

Let us now also calculate the specific radius at latitude ϕ, i.e. the length PC, labeled ρ in the diagram. From the equation of the ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

we substitute $x = \rho\cos(\phi')$ and $y = \rho\sin(\phi')$, and obtain:

$$\begin{aligned}
1 &= \frac{(\rho\cos(\phi'))^2}{a^2} + \frac{(\rho\sin(\phi'))^2}{b^2} \\
a^2b^2 &= \rho^2(b^2\cos^2(\phi') + a^2\sin^2(\phi')) \\
\therefore \rho &= \frac{ab}{\sqrt{b^2\cos^2(\phi') + a^2\sin^2(\phi')}} \qquad (7.5)
\end{aligned}$$

An alternate formula for ρ is given as follows:
From the equation of the ellipse and its derivative, substituting $1 - e^2$ for b^2/a^2,

$$\begin{cases} x^2 + \frac{y^2}{1-e^2} = a^2 \\ \frac{y}{x} = (1 - e^2)\tan(\phi) \end{cases}$$

we find:

$$
\begin{aligned}
y &= x(1-e^2)\tan(\phi) \\
\therefore x^2 + x^2(1-e^2)\tan^2(\phi) &= a^2 \\
\therefore x^2\left(1+(1-e^2)\cdot\frac{\sin^2(\phi)}{\cos^2(\phi)}\right) &= a^2
\end{aligned}
$$

$$
\begin{aligned}
\therefore x^2 &= \frac{a^2}{1+(1-e^2)\cdot\frac{\sin^2(\phi)}{\cos^2(\phi)}} \\
&= \frac{a^2\cos^2(\phi)}{\cos^2(\phi)+\sin^2(\phi)-e^2\sin^2(\phi)} \\
&= \frac{a^2\cos^2(\phi)}{1-e^2\sin^2(\phi)} \\
\therefore x &= \frac{a\cos(\phi)}{\sqrt{1-e^2\sin^2(\phi)}}
\end{aligned}
$$

And now solving for y:

$$
\begin{aligned}
y &= x(1-e^2)\tan(\phi) \\
&= \frac{a\cos(\phi)}{\sqrt{1-e^2\sin^2(\phi)}}\cdot(1-e^2)\cdot\frac{\sin(\phi)}{\cos(\phi)} \\
&= \frac{(1-e^2)a\sin(\phi)}{\sqrt{1-e^2\sin^2(\phi)}}
\end{aligned}
$$

Thus:

$$
x = \frac{a\cos(\phi)}{\sqrt{1-e^2\sin^2(\phi)}} \quad \text{and} \quad y = \frac{(1-e^2)a\sin(\phi)}{\sqrt{1-e^2\sin^2(\phi)}}
$$

But, $x = \rho\cos(\phi')$ and $y = \rho\sin(\phi')$, so:

$$
\left.
\begin{aligned}
\rho\cos(\phi') &= \frac{a\cos(\phi)}{\sqrt{1-e^2\sin^2(\phi)}} \\
\rho\sin(\phi') &= \frac{(1-e^2)a\sin(\phi)}{\sqrt{1-e^2\sin^2(\phi)}}
\end{aligned}
\right\} \tag{7.6}
$$

With the addition of an auxiliary ψ defined via $\sin(\psi) = e\sin(\phi) \therefore \sqrt{1-e^2\sin^2(\phi)} = \cos(\psi)$, these equations can be simplified:

$$
\left.
\begin{aligned}
\sin(\psi) &= e\sin(\phi) \\
\rho\cos(\phi') &= a\cos(\phi)\sec(\psi) \\
\rho\sin(\phi') &= a(1-e^2)\sin(\phi)\sec(\psi)
\end{aligned}
\right\} \tag{7.7}
$$

We can deduce from equation 7.7 by using the angle addition formulae

$$
\begin{aligned}
\rho\cos(\phi-\phi') &= \rho\cos(\phi)\cos(\phi') + \rho\sin(\phi)\sin(\phi') \\
&= a\cos^2(\phi)\sec(\psi) + a(1-e^2)\sin^2(\phi)\sec(\psi) \\
&= a\sec(\psi)(\cos^2(\phi)+\sin^2(\phi)-e^2\sin^2(\phi)) \\
&= a\sec(\psi)(1-e^2\sin^2(\phi)) \\
&= a\sec(\psi)(1-\sin^2(\psi)) \\
&= a\sec(\psi)\cos^2(\psi) \\
&= a\cos(\psi)
\end{aligned}
$$

and

$$\begin{aligned}\rho\sin(\phi-\phi') &= \rho\sin(\phi)\cos(\phi') - \rho\cos(\phi)\sin(\phi') \\ &= a\cos(\phi)\sec(\psi)\sin(\phi) - a(1-e^2)\sin(\phi)\sec(\psi)\cos(\phi) \\ &= a\cos(\phi)\sin(\phi)\sec(\psi)(1-(1-e^2)) \\ &= ae^2\cos(\phi)\sin(\phi)\sec(\psi)\end{aligned}$$

Thus:

$$\left.\begin{aligned}\rho\cos(\phi-\phi') &= a\cos(\psi) \\ \rho\sin(\phi-\phi') &= ae^2\cos(\phi)\sin(\phi)\sec(\psi)\end{aligned}\right\} \tag{7.8}$$

By multiplying the 2nd line of equation 7.7 with the 1st line of equation 7.8, we obtain:

$$\rho = a\sqrt{\frac{\cos(\phi)}{\cos(\phi')\cos(\phi-\phi')}} \tag{7.9}$$

Example 7.1 The Earth's equatorial radius and polar radius are given as 6378.137 km and 6356.752 km respectively. Find the radius of the Earth at geographic latitude $35°N$.

Solution
By equation 7.4:

$$\tan(\phi') = \frac{6356.752^2}{6378.137^2}\cdot\tan(35°)$$
$$\therefore \phi' = 34°\ 49'\ 9.79''.$$

Then by equation 7.5:

$$\begin{aligned}\rho &= \frac{ab}{\sqrt{b^2\cos^2(\phi')+a^2\sin^2(\phi')}} \\ &= \frac{6378.137\cdot 6356.752}{\sqrt{6356.752^2\cos^2(34°\ 49'\ 9.79'')+6378.137^2\sin^2(4°\ 49'\ 9.79'')}} \\ &= 6371.141 \text{ km.}\end{aligned}$$

The reader can verify that equation 7.9 gives the same answer.

7.4 The Normal

The normal line is the line PO, and is the true vertical line at point P. The distance PO, which we denote N, is given as follows. From the figure above, it is evident that:

$$CQ = N\cos(\phi) = \rho\cos(\phi')$$

and therefore,

$$N = \frac{\rho\cos(\phi')}{\cos(\phi)} = \frac{\frac{a\cos(\phi)}{\sqrt{1-e^2\sin^2(\phi)}}}{\cos(\phi)} = \frac{a}{\sqrt{1-e^2\sin^2(\phi)}} \tag{7.10}$$

or, by using the same auxiliary $\sin(\psi) = e\sin(\phi)$,

$$N = a\sec(\psi)$$

Let us now find the distance between the center and the intersection point of the normal line and the axis, i.e. the distance CO. From the triangle PMO and CMO, we can see:

$$\rho \sin(\phi - \phi') = CO \sin(90^\circ - \phi)$$

and therefore:

$$CO = \frac{\rho \sin(\phi - \phi')}{\cos(\phi)} \tag{7.11}$$

or, by 7.8:

$$CO = \frac{ae^2 \cos(\phi) \sin(\phi) \sec(\psi)}{\cos(\phi)} = ae^2 \sin(\phi) \sec(\psi) \tag{7.12}$$

Now we can properly account for altitude above sea level. To find the distance from the center of the earth to a point P' which is at a perpendicular altitude h above P, we first find the normal distance to the axis N and the center – intersection distance CO. Then, we add the altitude h to N, and then use the law of cosines to find the new geocentric distance D of the point P'.

This all can be summarized as:

$$D^2 = (N + h)^2 + CO^2 - 2 \cdot (N + h) \cdot CO \cdot \sin(\phi) \tag{7.13}$$

Example 7.2 Find the geocentric distance and latitude of a point with geographic latitude 35° and altitude 100 km.

Solution
From the last example, we know that for $\phi = 35^\circ$:

$$\phi' = 34^\circ\, 49'\, 9.79''$$
$$\rho = 6371.141 \text{ km}$$

Then, by equation 7.10:

$$N = \frac{6371.141 \cos(34^\circ\, 49'\, 9.79'')}{\cos(35^\circ)} = 6385.172 \text{ km}$$

And by equation 7.11:

$$CO = \frac{6371.141 \sin(35^\circ - 34^\circ\, 49'\, 9.79'')}{\cos(35^\circ)} = 24.518 \text{ km}$$

And now by equation 7.13:

$$D^2 = (6385.172 + 100)^2 + 24.518^2 - 2 \cdot (6385.172 + 100) \cdot 24.518 \cdot \sin(35^\circ) = 41\,875\,656 \text{ km}^2$$
$$\therefore D = 6471.140 \text{ km}$$

Now, by equation 7.10:

$$6385.172 + 100 = \frac{6471.140 \cos(\phi')}{\cos(35^\circ)}$$
$$\therefore \phi' = \arccos\left(\frac{6485.172 \cos(35^\circ)}{6471.140}\right)$$
$$= 34^\circ\, 49'\, 19.82''$$

7.5 The Curvature of the Earth

The radius of curvature R of a curve is given as:

$$R = \left| \frac{(1+y'^2)^{3/2}}{y''} \right|$$

And so for the Earth, which is an ellipse, we have from the equation of an ellipse:

$$\begin{aligned}
y' &= \frac{dy}{dx} = -\frac{b^2 x}{a^2 y} \\
y'' &= \frac{d^2y}{dx^2} = -\frac{b^2}{a^2 y} + \frac{b^2 x}{a^2 y^2}\frac{dy}{dx} \\
&= -\frac{b^2}{a^2 y} - \frac{b^2 x}{a^2 y^2}\frac{b^2 x}{a^2 y} \\
&= -\frac{b^2 a^2 y^2 - b^4 x^2}{a^4 y^3} \\
&= -\frac{b^2 a^2 y^2 - b^4 a^2 (1 - y^2/b^2)}{a^4 y^3} \\
&= -\frac{b^2 a^2 y^2 - b^4 a^2 - b^2 a^2 y^2}{a^4 y^3} \\
&= -\frac{b^4}{a^2 y^3}
\end{aligned}$$

And therefore we have:

$$\begin{aligned}
R &= \frac{(1 + \frac{b^4 x^2}{a^4 y^2})^{3/2}}{\frac{b^4}{a^2 y^3}} \\
&= \frac{(a^4 y^2 + b^4 x^2)^{3/2}}{a^6 y^3 \cdot \frac{b^4}{a^2 y^3}} \\
&= \frac{(a^4 y^2 + b^4 x^2)^{3/2}}{a^4 b^4}
\end{aligned} \tag{7.14}$$

When we substitute

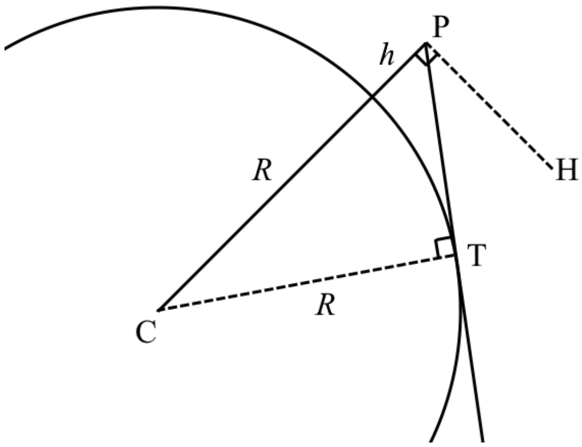
$$x = \frac{a\cos(\phi)}{\sqrt{1 - e^2 \sin^2(\phi)}} \quad \text{and} \quad y = \frac{(1-e^2)a\sin(\phi)}{\sqrt{1 - e^2\sin^2(\phi)}}$$

as well as $b^2 = a^2(1 - e^2)$ into equation 7.14, we obtain:

$$\begin{aligned}
R &= \frac{\left(a^4\left(\frac{(1-e^2)a\sin(\phi)}{\sqrt{1-e^2\sin^2(\phi)}}\right)^2 + b^4\left(\frac{a\cos(\phi)}{\sqrt{1-e^2\sin^2(\phi)}}\right)^2\right)^{3/2}}{a^4 b^4} \\
&= \frac{(a^4(1-e^2)^2 a^2 \sin^2(\phi) + b^4 a^2 \cos^2(\phi))^{3/2}}{a^4 b^4 (1 - e^2\sin^2(\phi))^{3/2}} \\
&= \frac{(b^4 a^2 \sin^2(\phi) + b^4 a^2 \cos^2(\phi))^{3/2}}{a^4 b^4 (1 - e^2\sin^2(\phi))^{3/2}} \\
&= \frac{b^6 a^3 (\sin^2(\phi) + \cos^2(\phi))^{3/2}}{a^4 b^4 (1 - e^2\sin^2(\phi))^{3/2}}
\end{aligned}$$

$$\begin{aligned} R &= \frac{b^6 a^3}{a^4 b^4 (1 - e^2 \sin^2(\phi))^{3/2}} \\ &= \frac{b^2}{a(1 - e^2 \sin^2(\phi))^{3/2}} \\ &= \frac{a^2(1 - e^2)}{a(1 - e^2 \sin^2(\phi))^{3/2}} \\ &= \frac{a(1 - e^2)}{(1 - e^2 \sin^2(\phi))^{3/2}} \end{aligned} \tag{7.15}$$

Locally therefore, the Earth can be approximated as a sphere with radius R. We can now calculate the altitude angle of the *true* horizon at height h above sea level.



In this diagram, the Earth has been approximated as a sphere with radius R. Note that the center of the approximating sphere C is the *center of curvature* and not the true center of the Earth. The center of curvature lies on the normal line OP at a distance $R + h$ from P.

The true horizon of an observer at the point P, which is at a height h above the Earth, is given by PT and not PH, which is called the *astronomical* or *geometric* horizon. The altitude angle of point T, HPT, called the *dip of the horizon* is therefore given by:

$$HPT = -(90° - CPT)$$

Where CPT is given by:

$$CPT = \arcsin\left(\frac{R}{R+h}\right)$$

Thus:

$$HPT = -\left(90° - \arcsin\left(\frac{R}{R+h}\right)\right) = -\arccos\left(\frac{R}{R+h}\right) \tag{7.16}$$

This approximation works best when the eccentricity of the ellipsoid and the altitude of the observer are low.

Example 7.3 Find the dip of the true horizon for an observer with geographic latitude 35° and altitude 100 km. Use $a = 6378.137$ km and $e = 0.081819$.

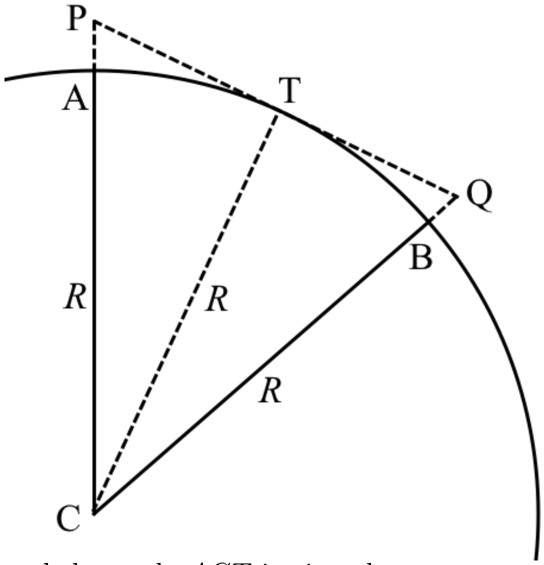
Solution
By equation 7.15, the radius of curvature of the Earth at this latitude is:

$$R = \frac{6378.137(1 - 0.081819^2)}{(1 - 0.081819^2 \sin^2(35°))^{3/2}} = 6356.427 \text{ km}$$

Therefore the dip angle of the horizon is (by equation 7.16):

$$\text{Dip of True Horizon } = -\arccos\left(\frac{6356.427}{6356.427 + 100}\right) = -10° \, 5' \, 50''$$

Thus, for example, sunrise will occur at point P when the Sun's altitude rises above $-10° \, 5' \, 50''$.



Due to the curvature of the Earth, objects on the Earth farther away from an observer will become invisible below the horizon to that observer.

For an observer at an altitude $AP = h$ above the surface of the Earth where the radius of curvature is R, for a point to be visible from a distance $AB = L$ away, it must be at least at altitude $BQ = z$ above the surface.

To calculate this distance, first we must find how far beyond B is from the horizon T. Evidently, the angle ACB is given (in radians) by:

$$ACB = \frac{L}{R} \tag{7.17}$$

and the angle ACT is given by:

$$ACT = \arccos\left(\frac{R}{R+h}\right) \tag{7.18}$$

Thus the angle TCB is given by:

$$TCB = ACB - ACT = \frac{L}{R} - \arccos\left(\frac{R}{R+h}\right)$$

Because $\cos(TCB) = R/(R+z)$, $z = R\sec(TCB) - R$ and therefore:

$$z = R\sec\left(\frac{L}{R} - \arccos\left(\frac{R}{R+h}\right)\right) - R \tag{7.19}$$

As with the previous example, the radius of curvature approximation works best when distances are small in comparison to the radius of the Earth.

Example 7.4 For an observer at latitude 35° and 5 meters above sea level, determine if a building 80 km away and 500 meters tall (500 m above sea level) would be visible.

Solution
In the last example we found that the radius of curvature for this latitude is 6356.427 km.
By equation 7.19, for a building to be visible from 80 kilometers away, it must be at least

$$z = 6356.427\sec\left(\frac{80}{6356.427} - \arccos\left(\frac{6356.427}{6356.427+0.005}\right)\right) - 6356.427 = 0.408 \text{ km} = 408 \text{ m}$$

tall. Therefore the top 92 meters of a building 500 meters tall would be visible.
Remember to carry out calculations using radians.

By rearranging equation 7.19, we can obtain:

$$L = R\arccos\left(\frac{R}{R+z}\right) + R\arccos\left(\frac{R}{R+h}\right) \tag{7.20}$$

Which gives the maximum distance away an object of height z above sea level would be visible to an observer at height h above sea level.

Example 7.5 Determine the maximum distance a building 500 meters tall (500 m above sea level) and located at latitude 35° would be visible to observers standing 5 meters above sea level.

Solution
Equation 7.20 gives:

$$L = 6356.427 \arccos\left(\frac{6356.427}{6356.427 + 0.5}\right) + 6356.427 \arccos\left(\frac{6356.427}{6356.427 + 0.005}\right) = 87.7 \text{ km}$$

Remember to carry out calculations using radians.

Note that this calculation of visible distance is not entirely accurate due to atmospheric effects. For example, Chicago is visible from across Lake Michigan due to the refraction of light by the air even though the curvature of the Earth would obscure it.

Chapter 8

Parallax

Our formulae for the location of planets in the sky, and hence the values in our ephemeris, are calculated for a hypothetical observer at the center of the Earth. These values are called the *true* values. On the surface, there is a measurable difference in the values of angles compared to their true values due to the difference in viewing location, particularly for close objects such as the Moon. The actual observed angles are called the *apparent* values, and their difference is called the *parallax*. Let us see how to calculate this parallax.

8.1 Apparent Equatorial Coordinates

We can formulate the position of any point on the Earth (p, q, s) in geocentric equatorial coordinates as follows:

$$\left.\begin{aligned} p &= \rho\cos(\phi')\cos(\Theta_L) \\ q &= \rho\cos(\phi')\sin(\Theta_L) \\ s &= \rho\sin(\phi') \end{aligned}\right\} \tag{8.1}$$

Where ρ is the distance to the geocenter, ϕ' is the geocentric latitude, and Θ_L is the local sidereal time, because Θ_L is the right ascension of the local meridian. Then, given that the true cartesian coordinates of a celestial body is (x, y, z), we can calculate its apparent cartesian coordinates (x', y', z') by:

$$(x', y', z') = (x - p, y - q, z - s) \tag{8.2}$$

Which can be turned back into spherical coordinates and then be used to calculate the phenomena detailed in chapter 6.
Naturally, if one is given the apparent coordinates then the true coordinates are:

$$(x, y, z) = (x' + p, y' + q, z' + s) \tag{8.3}$$

The differences in α and δ between the true and apparent values are called the parallax in right ascension and declension respectively.

While the disparity of location between the observer and the center of the Earth significantly changes the location of closer objects (like the Sun and the planets, and in particular the Moon) it does not matter much for very far objects like the stars, and the stars can be regarded has having 0 parallax, i.e. no difference in location whatsoever.

Example 8.1 On January 1, 2024 at standard time $00:00$ ($\Theta = 100°\ 9'\ 9.42''$), the true equatorial coordinates of the Moon were $\alpha = 10^h\ 35^m\ 11.55^s$ and $\delta = +12°\ 45'\ 8.3''$. Given that the distance to the Moon was $\Delta = 404\ 634.3$ km, calculate its apparent equatorial coordinates from $\phi = 35°$ and $l = 150°$.

Solution
In cartesian coordinates, the Moon's true coordinates were (by equation 1.1):

$$
\begin{aligned}
x &= 404\,634.3\cos(12^\circ\,45'\,8.3'')\cos(10^h\,35^m\,11.55^s) &= -367940.1\\
y &= 404\,634.3\cos(12^\circ\,45'\,8.3'')\sin(10^h\,35^m\,11.55^s) &= 142728.4\\
z &= 404\,634.3\sin(12^\circ\,45'\,8.3'') &= 89317.63
\end{aligned}
$$

The local sidereal time was (by equation 5.3):

$$\Theta_L = 100^\circ\,9'\,9.42'' + 150^\circ = 250^\circ\,9'\,9.42''$$

The radius of the Earth at $\phi = 35^\circ$ ($\phi' = 34^\circ\,49'\,9.79''$) was calculated in example 7.1 to be 6371.141 km. Thus, the equatorial coordinates of the point $\phi = 35^\circ, l = 150^\circ$ is (by equation 8.1):

$$
\begin{aligned}
p &= 6371.141\cos(34^\circ\,49'\,9.79'')\cos(250^\circ\,9'\,9.42'') &= -1775.813\\
q &= 6371.141\cos(34^\circ\,49'\,9.79'')\sin(250^\circ\,9'\,9.42'') &= -4919.741\\
s &= 6371.141\sin(34^\circ\,49'\,9.79'') &= 3637.867
\end{aligned}
$$

Thus, by equation 8.2:

$$
\begin{aligned}
x' &= -367940.1 - (-1775.813) &= -366164.3\\
y' &= 142728.4 - (-4919.741) &= 147648.1\\
z' &= 89317.63 - 3637.867 &= 85679.76
\end{aligned}
$$

Thus, by equation 1.2:

$$
\begin{aligned}
\alpha' &= \arctan(147648.1, -366164.3) &= 10^h\,32^m\,9.43^s\\
\delta' &= \arcsin\left(\frac{85679.76}{\sqrt{(-366164.3)^2 + 147648.1^2 + 85679.76^2}}\right) &= +12^\circ\,14'\,38.9''
\end{aligned}
$$

In a worldbuilding setting however, the cartesian coordinates of the Moon would already be known, and therefore one can skip the first step.

To demonstrate the difference, let's calculate the horizontal coordinates of the Moon.

Example 8.2 Calculate the horizontal coordinates of the Moon at the time and location from the previous example using the true and apparent coordinates.

Solution
We use equation 6.3 with the geodetic latitude 35°. For the true coordinates:

$$
\begin{aligned}
h &= 250^\circ\,9'\,9.42'' - 10^h\,35^m\,11.55^s = 91^\circ\,21'\,16.17''\\
\delta &= 12^\circ\,45'\,8.3''
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
A &= 281^\circ\,15'\,18.12''\\
a &= 6^\circ\,11'\,2.82''
\end{aligned}
$$

For the apparent coordinates:

$$
\begin{aligned}
h' &= 250^\circ\,9'\,9.42'' - 10^h\,32^m\,9.43^s = 92^\circ\,6'\,48.03''\\
\delta' &= 12^\circ\,14'\,38.9''
\end{aligned}
$$

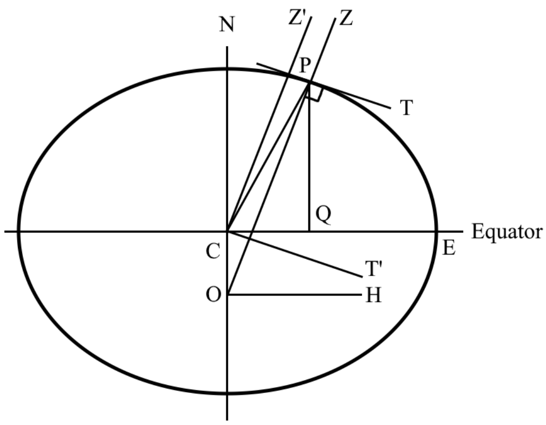
Therefore:

$$A' = 281^\circ\, 15'\, 28.20''$$
$$a' = 5^\circ\, 17'\, 8.62''$$

Comparing to the real life values of $A = 281^\circ, a = 5^\circ$ (rounded values), it is evident that the values calculated using the apparent coordinates are what an observer standing at that location would actually see. Thus, the notion of "geocentric (true) horizontal coordinates" is merely a simplification and has no real-life meaning. That is not to say it is not useful, as we will see in the next section.

8.2 Apparent Horizontal Coordinates

It is possible to calculate the apparent horizontal coordinates from the geocentric horizontal coordinates, thus removing the need to calculate the apparent right ascension and declension.



Consider a coordinate system where the origin is the place of observation P, the x-axis points northward along the meridian along the horizon, i.e. TP, and the z-axis points upward towards the Zenith, i.e. PZ. The y-axis is then the East-West line at point P. In this coordinate system, the position (x', y', z') of an object in the sky is given as (by equation 1.1):

$$x' = \Delta' \cos(a') \cos(A')$$
$$y' = \Delta' \cos(a') \sin(A')$$
$$z' = \Delta' \sin(a')$$

Where Δ' is the apparent distance to the object. These are the apparent horizontal coordinates of the object.

Consider another coordinate system, parallel to the first but centered at the Earth's center C, i.e. the x-axis points in the direction $T'C$, and the z-axis points in the direction CZ'. The position (x, y, z) of the object is given in this new coordinate system as (by equation 1.1):

$$x = \Delta \cos(a) \cos(A)$$
$$y = \Delta \cos(a) \sin(A)$$
$$z = \Delta \sin(a)$$

These are the true (geocentric) horizontal coordinates of the object.
The position (p, q, s) of P in the second (geocentric) system are given as:

$$\left.\begin{aligned} p &= -\rho \sin(\phi - \phi') \\ q &= 0 \\ s &= \rho \cos(\phi - \phi') \end{aligned}\right\} \quad (8.4)$$

Because $\phi - \phi' = CPO = PCZ'$. The minus sign on p is due to the fact that the line PZ is located closer to the equator than CZ', i.e. more south in the Northern hemisphere and more north in the Southern hemisphere. Because the x-axis points northward, there must be a negative sign.

From equation 8.4, the apparent horizontal coordinates can be described by equation 8.2.

Example 8.3 Calculate the apparent horizontal coordinates of the Moon from $\phi = 35°$, given that the geocentric horizontal coordinates of the Moon at this time for this latitude was $\Delta = 404$ 634.3 km, $A = 281°\ 15'\ 18.12'', a = 6°\ 11'\ 2.82''$.

Solution
The Moon's coordinates by equation 1.1:

$$
\begin{aligned}
x &= 404\,634.3\cos(6°\ 11'\ 2.82'')\cos(281°\ 15'\ 18.12'') &&= 78515.54 \\
y &= 404\,634.3\cos(6°\ 11'\ 2.82'')\sin(281°\ 15'\ 18.12'') &&= -394543.1 \\
z &= 404\,634.3\sin(6°\ 11'\ 2.82'') &&= 43588.73
\end{aligned}
$$

We calculated in previous examples that for this latitude:

$$
\begin{aligned}
\rho &= 6371.141 \text{ km} \\
\phi' &= 34°\ 49'\ 9.79'' \\
\therefore \phi - \phi' &= 10'\ 50.21''
\end{aligned}
$$

Thus, by equation 8.4:

$$
\begin{aligned}
p &= -6371.141\sin(10'\ 50.21'') = -20.08377 \\
q &= 0 \\
s &= 6371.141\cos(10'\ 50.21'') \quad = 6371.109
\end{aligned}
$$

Thus, by equation 8.2:

$$
\begin{aligned}
x' &= 78515.54 - (-20.08377) &&= 78535.46 \\
y' &= -394543.1 - 0 &&= -394543.1 \\
z' &= 43588.73 - 6371.109 &&= 37217.62
\end{aligned}
$$

Thus, by equation 1.2:

$$
\begin{aligned}
A' &= \arctan(-394543.1, 78535.46) &&= 281°\ 15'\ 28.14'' \\
a' &= \arcsin\left(\frac{37217.62}{\sqrt{78535.46^2 + (-394543.1)^2 + 37217.62^2}}\right) &&= 5°\ 17'\ 8.60''
\end{aligned}
$$

Which agree with example 8.2 within rounding error.

8.3 Parallax Corrected Sunrise Equation

In chapter 6, we derived the sunrise equation that can be used to determine the hour angle for when an object would be at the horizon. However, our formulation used the geocentric horizontal coordinates which are not what an observer on the surface would actually see. To calculate the observed hour angle, we must correct for parallax.

In order for the apparent altitude to be $0°$, z' must be equal to 0, and thus by equation 8.2, z must equal s. z, the geocentric horizontal z coordinate of the object is given by equation 6.3:

$$z = \Delta\cos(\phi)\cos(\delta)\cos(h) + \Delta\sin(\phi)\sin(\delta)$$

and s is given by equation 8.4:

$$s = \rho \cos(\phi - \phi')$$

Equating the two and solving for $\cos(h)$ yields the parallax corrected sunrise equation:

$$\cos(h) = \frac{\rho \cos(\phi - \phi')}{\Delta \cos(\phi) \cos(\delta)} - \tan(\phi) \tan(\delta) \tag{8.5}$$

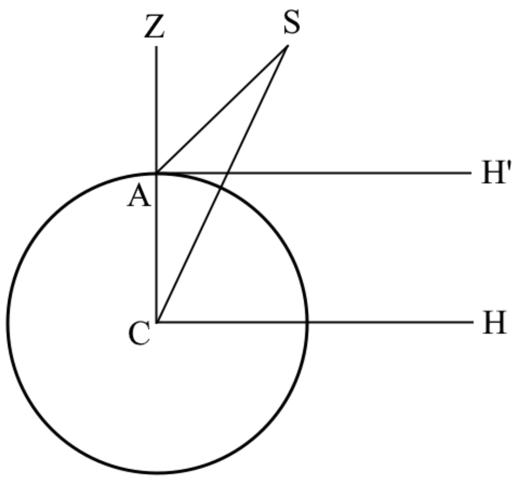
This formula only involves geocentric coordinates of the celestial object.

It is easy to see that if $\rho = 0$, i.e. the place of observation is the center of the Earth, this equation reduces to the regular sunrise equation. It is evident from the many examples of chapter 6 that the uncorrected version is accurate enough for most things: obviously the stars (since they have no parallax), and even for closer objects like the Sun. However, even closer objects like the Moon can have a very significant parallax (see the next section) and therefore the parallax corrected version of the sunrise equation should be used.

An interesting thing happens when Δ is small enough (within a few planetary radii) and δ is small enough: both North and South poles may not be able to see the object above the horizon (i.e. $\cos(h) > 1$ for both poles). However, never will both poles see the object above the horizon. This makes sense geometrically.

Also note that the parallax corrected sunrise equation still does not account for the apparent size of the object.

8.4 Equatorial Horizontal Parallax



In this diagram, the place of observation is A, the object being observed is S, and the geocentric and local horizons are given by CH and AH' respectively. The true altitude of the star is SCH and the apparent altitude is SAH'. Simple geometry will show that:

$$SAH' - SCH = ASC$$

Or in other words, the parallax in altitude is equal to the angle subtended by the radius of the Earth from S. It is evident therefore that maximum parallax is achieved when S is at the local horizon H', when S subtends the whole radius of the Earth. This angle is called the *horizontal parallax*, and is maximum when the radius of the Earth CA is the equatorial radius, in which case it is called the *equatorial horizontal parallax* (denoted by π).

It is evident then, that:

$$\sin(\pi) = \frac{a}{\Delta} \tag{8.6}$$

Where a is the equatorial radius of the Earth and Δ is the geocentric distance to the object.

Example 8.4 Determine the maximum possible parallax in the position of the Moon given that the Moon was 404 634.3 km away and that the equatorial radius of the Earth is 6378.137 km.

Solution

The maximum possible parallax is the equatorial horizontal parallax, so by equation 8.6:

$$\sin(\pi) = \frac{6378.137}{404\,634.3}$$
$$\therefore \pi = 54'\,11''$$

Thus the position of the Moon could vary by up to 54′ 11″ from the geocentric position depending on the location of the observer on the Earth.

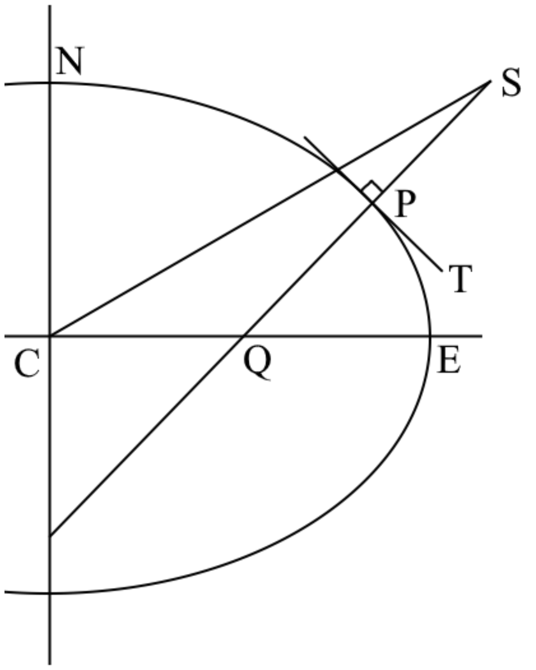
The absolute maximum parallax of the Moon happens when the Moon is closest to the Earth. The distance from the Earth to the Moon in this case is 356 400 km, therefore:

$$\pi = \arcsin\left(\frac{6378.137}{356\,400}\right) = 1^\circ\,1'\,32''$$

The Moon's position could vary from the geocentric position by more than a degree in this case.

8.5 Parallax Corrected Subsolar Point

As we saw in the last section, the parallax is maximum if the object is on the horizon. Similarly, the parallax is minimum if the object is at the Zenith, so much so that on a spherical Earth, no parallax exists for an object at the Zenith. Even for an ellipsoidal Earth, the parallax in the longitude of the subsolar point is zero, and the parallax in the latitude of the subsolar point is so miniscule for a low flattening value that it is not worth calculating. However, the method of calculation is given here regardless.



Consider a meridional slice of an ellipsoidal Earth where N is the North pole and CE is the line of the equator. S is the position of an object a geocentric distance Δ and geocentric declination $SCE = \delta$. The point P is the sub-object point, where the object appears perpendicular to the local horizon PT (i.e. at the Zenith). The angle we are interested in is the angle PQE, the geodetic latitude of P.

Let's define a 2D coordinate frame where the C is the origin, the x-axis points towards E, and the y-axis points towards N. Let the coordinates of P in this system be (x_0, y_0) and coordinates of S be (v, u). Then, by geometry:

$$v = \Delta\cos(\delta)$$
$$u = \Delta\sin(\delta)$$

Also, as found earlier (in section 7.2), the derivative of equation of the ellipse is given by:

$$\frac{y}{x} = -\frac{b^2}{a^2}\frac{dx}{dy}$$

Therefore the slope m of the normal line at a point (x_0, y_0) is given by:

$$m = -\frac{dx}{dy} = \frac{a^2 y_0}{b^2 x_0}$$

Plugging m into the equation of a line passing through (x_0, y_0), we obtain:

$$\begin{aligned} y - y_0 &= \frac{a^2 y_0}{b^2 x_0}(x - x_0) \\ \therefore b^2 x_0 (y - y_0) &= a^2 y_0 (x - x_0) \\ \therefore b^2 y x_0 - b^2 x_0 y_0 &= a^2 x y_0 - a^2 y_0 x_0 \end{aligned}$$

Dividing both sides by $x_0 y_0$ we obtain:

$$\frac{b^2 y}{y_0} - b^2 = \frac{a^2 x}{x_0} - a^2$$

$$\therefore 0 = \frac{a^2 x}{x_0} - \frac{b^2 y}{y_0} - a^2 + b^2 \tag{8.7}$$

At this point it is convenient to introduce an auxiliary variable θ such that:

$$\left.\begin{aligned} x_0 &= a\cos(\theta) \\ y_0 &= b\sin(\theta) \end{aligned}\right\} \tag{8.8}$$

The reader can verify that this does indeed satisfy the equation for an ellipse (equation 7.3).
Note that θ does not represent any real angle, it is just a convenient auxiliary that parametrizes the ellipse. By plugging equations 8.8 into equation 8.7, the equation for the normal at (x_0, y_0) is given by:

$$ax\sec(\theta) - by\csc(\theta) - a^2 + b^2 = 0$$

This normal line must pass through $S(v, u)$. Thus, we can substitude x and y by $v = \Delta\cos(\delta)$ and $u = \Delta\sin(\delta)$:

$$a\Delta\cos(\delta)\sec(\theta) - b\Delta\sin(\delta)\csc(\theta) - a^2 + b^2 = 0 \tag{8.9}$$

Solving this equation for θ gives the coordinates of P by equation 8.8, which gives the geocentric latitude by $\tan(\phi') = y_0/x_0$, which finally gives the geodetic latitude by equation 7.4. Equation 8.9 is not easy to solve however, and the best way to solve it is by numerical approximation: the derivative is given here for use in Newton Raphson iteration.

$$a\Delta\cos(\delta)\tan(\theta)\sec(\theta) + b\Delta\sin(\delta)\cot(\theta)\csc(\theta) \tag{8.10}$$

Using $\theta = \delta$ is a good first guess. Note that all angles must be reckoned in radians for Newton Raphson's method.
Also note that as the flattening increases, and δ gets closer to 0, the rate of convergence gets slower and slower and more and more iterations are needed, to the point where bisection might be better. However this should not be a problem for most planets as they have very low flatenning values.

Example 8.5 Given the true declension of the Moon $\delta = 12^\circ\ 45'\ 8.3''$ and the geocentric distance $\Delta = 404\ 634.3$ km, find the parallax corrected latitude of the sublunar point.
Use $a = 6378.137$ km and $b = 6356.752$ km.

Solution
δ in radians is 0.2225697 rad. We can now perform Newton iteration:

$$\theta_{n+1} = \theta_n - \frac{a\Delta\cos(\delta)\sec(\theta_n) - b\Delta\sin(\delta)\csc(\theta_n) - a^2 + b^2}{a\Delta\cos(\delta)\tan(\theta_n)\sec(\theta_n) + b\Delta\sin(\delta)\cot(\theta)\csc(\theta_n)} \tag{8.11}$$

Using $\theta_1 = \delta$ as a first guess, this gives $\theta_2 = 0.2218684$ rad. The table of repetitions is given here:

n	θ_n
1	0.2225697
2	0.2218684
3	0.2218704
4	0.2218704

The geocentric latitude of P is given then as (by equation 8.8):

$$\begin{aligned} x_0 &= 6378.137\cos(0.2218704) &= 6221.793 \\ y_0 &= 6356.752\sin(0.2218704) &= 1398.832 \\ \therefore \phi' &= \arctan\left(\frac{1398.832}{6221.793}\right) &= 12^\circ\ 40'\ 15.6'' \end{aligned}$$

Therefore the geodetic latitude of P is (by the reverse of equation 7.4):

$$\phi = \arctan\left(\frac{6378.137^2}{6356.752^2}\tan(12^\circ\ 40'\ 15.6'')\right) = 12^\circ\ 45'\ 13.0''$$

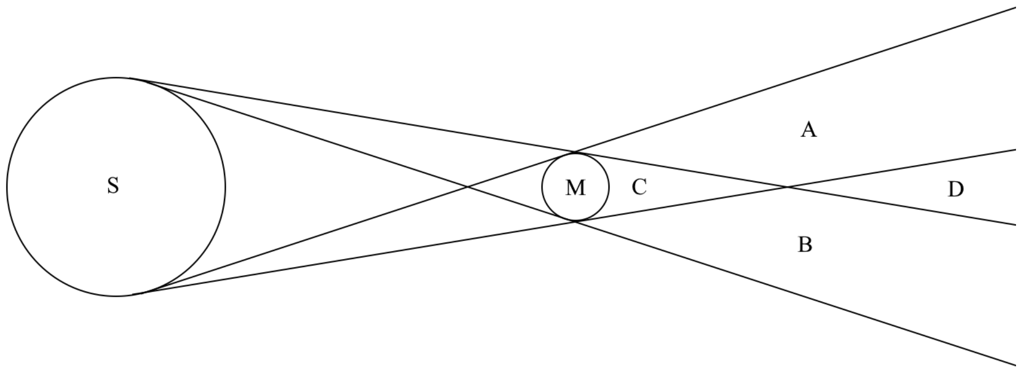
Which is less than a 5 arcsecond difference from the value given by the simple $\phi = \delta$. With a more significant flattening value this difference becomes more appreciable.

Chapter 9

Solar Eclipses

This chapter will go into detail about solar eclipses, and end with a brief explanation on lunar occultations of planets and stars. We follow Chauvenet (1891), Seidelmann (1992), and Green (1985).

9.1 Partial, Total, and Annular Solar Eclipses

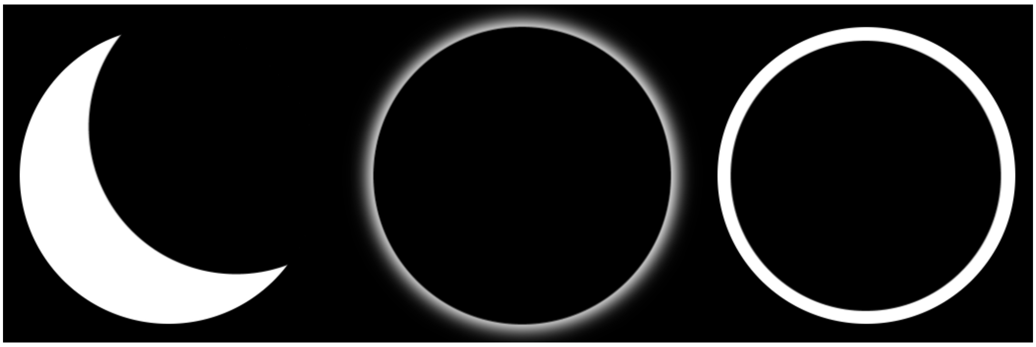


Any object illuminated by a non-point source casts two shadows: the *penumbra* and the *umbra*. In the diagram above, the penumbral shadow cast by the Moon (M) is shown by regions A and B. An observer in these regions sees the Moon slightly offset from the Sun (S) and so only sees a part of the Sun covered by the Moon. This is known as a *partial solar eclipse* (see leftmost figure below).

The umbral region is marked by regions C and D. An observer in region C cannot see the Sun at all since it is completely blocked by the Moon. This is known as a *total solar eclipse* (usually, the corona (the outer atmosphere) of the Sun is still visible; see middle figure below). In region D (technically called the *antumbra*), the Moon still coveres the center of the Sun but does not appear large enough to cover all of it. Therefore the Sun looks like a glowing ring around the Moon, and therefore this is known as an *annular solar eclipse* (see rightmost figure below).

On our Earth, our Moon is just at the perfect distance where the orbital eccentricity allows the Moon to be both close enough and far enough to create all three partial, total, and annular solar eclipses (and sometimes a total eclipse becomes annular halfway through or vice versa, this is known as a *hybrid solar eclipse*). This is not true for the vast majority of planets: Mars with its two small moons only experiences annular and partial solar eclipses.

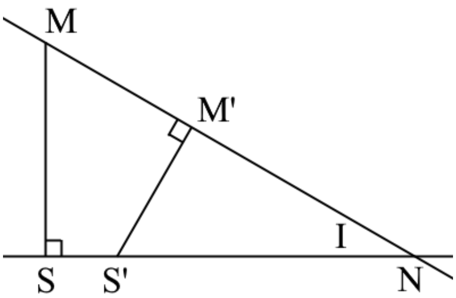
Total, annular, and hybrid eclipses are collectively called *umbral* or *central eclipses*. (Technically, central eclipses only refer to eclipses where the center of the shadow falls on the Earth. However, because the umbra is so small, preactically all umbral eclipses are central. This may not be true in rare cases, or if the umbra is larger.)



Left: Partial Solar Eclipse. Center: Total Solar Eclipse. Right: Annular Solar Eclipse.

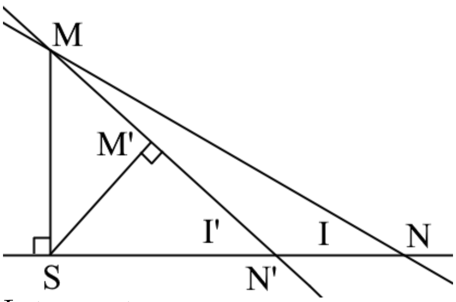
9.2 Conditions for Eclipse

A solar eclipse can only happen if the Moon and the Sun are in the sky in the same position, i.e. only when they are in conjunction, a.k.a. new Moon. Not all new moons result in a solar eclipse however, due to the inclination of the Moon's orbit to the Ecliptic.



Consider this diagram. In this diagram, the line SN is the ecliptic and MN is the orbit of the Moon, and therefore N is the lunar orbital (descending) node. Therefore the Moon moves along MN and the Sun moves along SN at a slower pace.

Say M and S are the positions of the Moon and Sun respectively at the time of ecliptic conjunction, and M' and S' are the positions of the Moon and Sun at their least true separation.



However, an easier way of looking at this is by fixing the Sun in one place and considering the *relative motion* of the Moon to the Sun.

In this diagram, the Sun has been fixed in place, and therefore the Moon now travels along an altered orbit MN'. M' is still the position of the Moon at its least true separation with the Sun.

Let us put

$$\begin{aligned}\beta &= \text{ The Moon's ecliptic latitude at conjunction } = SM\\ \Sigma &= \text{ The least true separation between the Sun and Moon } = SM'\\ I &= \text{ The inclination of the Moon's orbit to the ecliptic } = SNM\\ I' &= \text{ The inclination of the Moon's orbit to the ecliptic keeping the Sun fixed } = SN'M\end{aligned}$$

Then, by geometry we have $MSM' = I'$. Therefore:

$$\Sigma = \beta \cos(I') \tag{9.1}$$

But also,

$$\tan(I') = \frac{SN}{SN'} \tan(I)$$

If we put

$$
\begin{aligned}
\sigma &= \text{ The Sun's motion in longitude} \\
\mu &= \text{ The Moon's motion in longitude} \\
t &= \text{ The time it takes for the Moon to reach the node}
\end{aligned}
$$

Then,

$$
\begin{aligned}
SN &= \mu t \\
SN' &= (\mu - \sigma)t \\
\therefore \frac{SN}{SN'} &= \frac{\mu t}{\mu t - \sigma t}
\end{aligned}
$$

Dividing both the numerator and denominator by σt and setting $\mu/\sigma = q$, we get:

$$
\begin{aligned}
\frac{SN}{SN'} &= \frac{q}{q-1} \\
\therefore \tan(I') &= \frac{q}{q-1} \tan(I) \qquad (9.2)
\end{aligned}
$$

The apparent least true separation from somewhere on the surface of the Earth may differ from Σ however by up to the difference in parallax of the two bodies, so if we put:

$$
\begin{aligned}
\pi &= \text{ The Moon's equatorial horizontal parallax} \\
\pi' &= \text{ The Sun's equatorial horizontal parallax}
\end{aligned}
$$

we get:

$$\text{least apparent separation } = \Sigma - (\pi - \pi')$$

An eclipse will occur if this least apparent separation is less than the sum of the apparent radii of the Sun and the Moon (i.e. the Sun and the Moon overlap). If we put:

$$
\begin{aligned}
s &= \text{ The Moon's apparent radius} \\
s' &= \text{ The Sun's apparent radius}
\end{aligned}
$$

We have in the case of eclipse:

$$\Sigma - (\pi - \pi') < s + s'$$

or:

$$\beta < (s + s' + \pi - \pi') \sec(I') \qquad (9.3\text{-i})$$

This is the condition for solar eclipse in general. Following the same logic, a central eclipse will occur if:

$$\beta < |s - s' + \pi - \pi'| \sec(I') \qquad (9.3\text{-ii})$$

This means that if $(s - s' + \pi - \pi') \sec(I') < \beta < (s + s' + \pi - \pi') \sec(I')$, a solar eclipse will occur, but it will be partial everywhere on the Earth.

Example 9.1 Determine if the new Moon of April, 2024 will result in a solar eclipse.

Solution
We can find via the method of example 4.4 that the new Moon occured on April 8, 2024 at 18 : 21. At this time:

$$\begin{aligned} \lambda_{\text{Moon}} = \lambda_{\text{Sun}} &= 19^\circ\, 4'\, 12.19'' \\ \beta_{\text{Moon}} &= 0^\circ\, 20'\, 52.53'' \\ \Delta_{\text{Moon}} &= 359\,807.95 \text{ km} \\ \Delta_{\text{Sun}} &= 149\,823\,425.56 \text{ km} \end{aligned}$$

Also:

$$\begin{aligned} \text{Earth's equatorial radius } &= 6378.137 \text{ km} \\ \text{Moon's radius } &= 1737.4 \text{ km} \\ \text{Sun's radius } &= 696\,000 \text{ km} \\ I &= 5.14^\circ \end{aligned}$$

Thus by equations 4.1 and 8.6:

$$\begin{aligned} s &= \arcsin\left(\frac{1737.4}{359\,807.95}\right) &&= 16'\, 35.99'' \\ s' &= \arcsin\left(\frac{696\,000}{149\,823\,425.56}\right) &&= 15'\, 58.2'' \\ \pi &= \arcsin\left(\frac{6378.137}{359\,807.95}\right) &&= 1^\circ\, 0'\, 56.55'' \\ \pi' &= \arcsin\left(\frac{6378.137}{149\,823\,425.56}\right) &&= 8.78'' \end{aligned}$$

To determine q, we need σ and μ, the derivatives of longitude of the Sun and Moon. We take one hour as the time step.
At 19 : 21:

$$\begin{aligned} \lambda_{\text{Moon}} &= 19^\circ\, 41'\, 40.88'' \\ \lambda_{\text{Sun}} &= 19^\circ\, 6'\, 34.7'' \end{aligned}$$

Therefore:

$$\begin{aligned} \mu &= \frac{19^\circ\, 41'\, 40.88'' - 19^\circ\, 4'\, 12.19''}{1h} = 37.48'/h \\ \sigma &= \frac{19^\circ\, 6'\, 34.7'' - 19^\circ\, 4'\, 12.19''}{1h} = 2.38'/h \end{aligned}$$

Therefore:

$$\begin{aligned} q &= \frac{37.48}{2.38} = 15.7479 \\ \therefore I' &= \arctan\left(\frac{15.7479}{15.7479 - 1}\tan(5.14^\circ)\right) = 5.486^\circ \end{aligned}$$

Therefore:

$$\begin{aligned} (s + s' + \pi - \pi')\sec(I') &= (16'\, 35.99'' + 15'\, 58.2'' + 1^\circ\, 0'\, 56.55'' - 8.78'')\sec(5.486^\circ) \\ &= 1^\circ\, 33'\, 47.73'' \end{aligned}$$

Which is greater than β_{Moon} at conjunction. Therefore, a solar eclipse will occur. Also:

$$\begin{aligned}(s - s' + \pi - \pi')\sec(I') &= (16'\,35.99'' - 15'\,58.2'' + 1^\circ\,0'\,56.55'' - 8.78'')\sec(5.486^\circ) \\ &= 1^\circ\,1'\,42.52''\end{aligned}$$

Which is greater than β_{Moon} at conjunction. Therefore, a central eclipse will occur.

There is a shorter way of doing this: we can precalculate the minimum possible value of $(s \pm s' + \pi - \pi')\sec(I')$, and if β_{Moon} at conjunction is less than this minimum value, a solar eclipse must surely occur. We can also precalculate the maximum possible value of $(s \pm s' + \pi - \pi')\sec(I')$, and if β_{Moon} at conjunction is greater than this maximum value, a solar eclipse will surely not occur. If it is in between these two values, the full calculation is required.

9.3 Orientation and Position of the Shadow and the Observer

The shadow of the Moon points in the direction of the Moon from the Sun. Thus, if we say that:

$$\begin{aligned}v, u, w &= \text{The geocentric equatorial cartesian coordinates of the Moon} \\ v', u', w' &= \text{The geocentric equatorial cartesian coordinates of the Sun}\end{aligned}$$

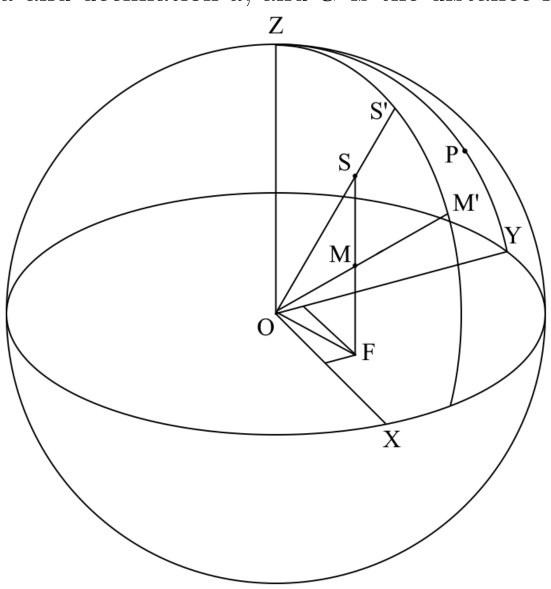
We have for the coordinates of the Sun from the Moon:

$$(v' - v, u' - u, w' - w)$$

And thus if we convert these to spherical coordinates:

$$\left.\begin{aligned}G\cos(d)\cos(a) &= v' - v \\ G\cos(d)\sin(a) &= u' - u \\ G\sin(d) &= w' - w\end{aligned}\right\} \quad (9.4)$$

We have the direction of the shadow of the Moon in equatorial spherical coordinates as right ascension a and declination d, and G is the distance from the Moon to the Sun.



Consider this diagram. Here, O is the center of the Earth, P is the celestial North pole, and M and S are the true positions of the center of the Moon and Sun. M' and S' are their positions projected on to the celestial sphere. Thus, SM is the shadow. Because the celestial sphere is infinite (of arbitrary radius), we can say that the shadow originates from the infinitely far away vanishing point Z, a point in the direction of SM.

We define a coordinate system as such: Let the origin be the center of the Earth, the z-axis point in the direction Z, the y-axis perpendicular to the z-axis such that it points towards North, and the x axis perpendicular to both such that it points to the point on the celestial equator with right ascension $a + 90^\circ$.

In this coordinate system, the xy-plane is known as the *fundamental plane*, and the point at which the shadow meets the fundamental plane is F.

To convert from the equatorial coordinate frame to the fundamental frame, we first rotate the equatorial frame by $a + 90°$ about the z-axis to align the x-axis:

$$R_1 = \begin{bmatrix} \cos(a+90°) & \sin(a+90°) & 0 \\ -\sin(a+90°) & \cos(a+90°) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then we rotate by $90° - d$ about the x-axis to align the z-axis with the shadow while keeping the y-axis pointed towards North.

$$R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(90°-d) & \sin(90°-d) \\ 0 & -\sin(90°-d) & \cos(90°-d) \end{bmatrix}$$

The total rotation is the product of the two matrices:

$$R = R_2 R_1 = \begin{bmatrix} -\sin(a) & \cos(a) & 0 \\ -\cos(a)\sin(d) & -\sin(a)\sin(d) & \cos(d) \\ \cos(a)\cos(d) & \sin(a)\cos(d) & \sin(d) \end{bmatrix} \tag{9.5}$$

We can now express the coordinates of both the Moon and observer in this new system. Let:

$x, y, z =$ The geocentric fundamental cartesian coordinates of the Moon
$\xi, \eta, \zeta =$ The geocentric fundamental cartesian coordinates of the observer

Then:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = R \begin{bmatrix} v \\ u \\ w \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = R \begin{bmatrix} \rho\cos(\phi')\cos(\Theta_L) \\ \rho\cos(\phi')\sin(\Theta_L) \\ \rho\sin(\phi') \end{bmatrix} \tag{9.6}$$

Where ρ is the radius of the Earth at geocentric latitude ϕ'.
Let's expand the second of these equations (using angle addition formulae to simplify):

$$\begin{aligned} \xi &= \rho\cos(\phi')\sin(\Theta_L - a) \\ \eta &= \rho[\sin(\phi')\cos(d) - \cos(\phi')\sin(d)\cos(\Theta_L - a)] \\ \zeta &= \rho[\sin(\phi')\sin(d) + \cos(\phi')\cos(d)\cos(\Theta_L - a)] \end{aligned}$$

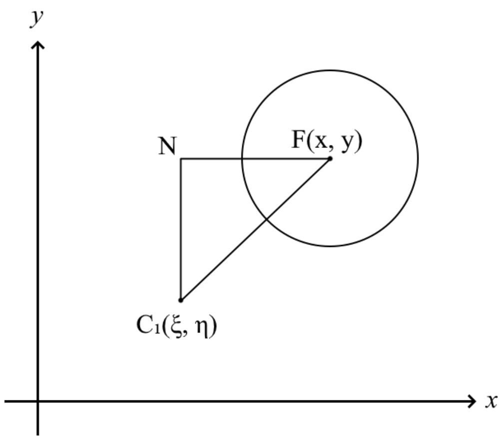
But $\Theta_L - a$ is the local hour angle of the point Z (the point pointing in the direction of the shadow), which we will denote by θ. Therefore:

$$\left.\begin{aligned} \xi &= \rho\cos(\phi')\sin(\theta) \\ \eta &= \rho[\sin(\phi')\cos(d) - \cos(\phi')\sin(d)\cos(\theta)] \\ \zeta &= \rho[\sin(\phi')\sin(d) + \cos(\phi')\cos(d)\cos(\theta)] \end{aligned}\right\} \tag{9.7}$$

θ is easily calculable (by equation 6.2) as just the hour angle of the shadow axis at the standard meridian (denoted by μ) plus the longitude of the place of observation (which we will denote by λ).

The inverse transformation is given by the transpose of R:

$$R^{-1} = R^T = \begin{bmatrix} -\sin(a) & \cos(a)\sin(d) & \cos(a)\cos(d) \\ \cos(a) & \sin(a)\sin(d) & \sin(a)\cos(d) \\ 0 & \cos(d) & -\sin(d) \end{bmatrix} \tag{9.8}$$



On the fundamental plane, the situation looks like this diagram: where the shadow of the Moon is centered on F, the projection of (x, y, z) onto the fundamental plane, and the observer is at C_1, the projection of (ξ, η, ζ) on the fundamental plane.

The distance between the shadow and the observer $\Delta = FC_1$ is given by:

$$\Delta^2 = (x - \xi)^2 + (y - \eta)^2 \tag{9.9}$$

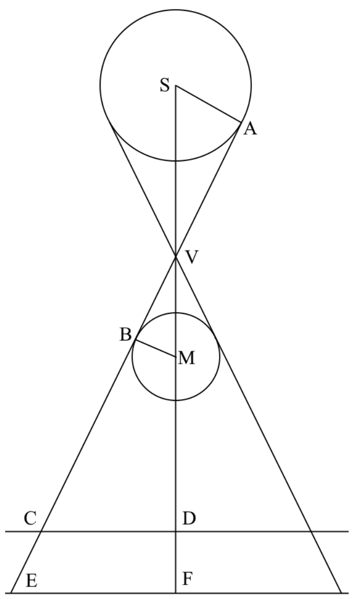
Which can also be expressed trigonometrically, which will be useful later:

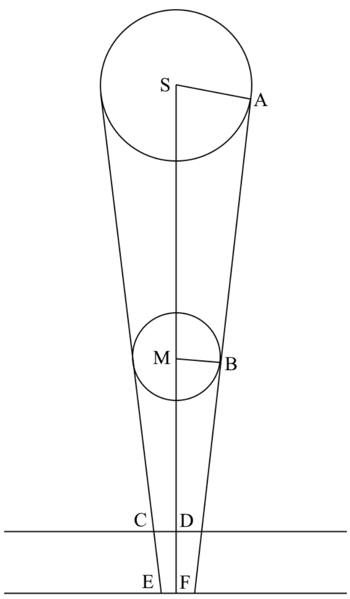
$$\left.\begin{aligned} \Delta \sin(Q) &= x - \xi \\ \Delta \cos(Q) &= y - \eta \end{aligned}\right\} \tag{9.10}$$

Where Q is the angle NC_1F. Notice that Q describes the position angle of the shadow in relation to the observer, and so we can generalize Q to cover the full range from 0° to 360° via equation 9.10. From these equations we can also see that if $\Delta \cos(Q) > 0$, $y - \eta > 0$ and thus η must be less than y, i.e. the observer is to the South of the shadow. Similarly, if $\Delta \cos(Q) < 0$, then $y - \eta < 0$ and so the observer must be North of the shadow.

9.4 The Size of the Shadow

In these diagrams, S is the center of the Sun, M is the center of the Moon, the line EF is the fundamental plane, and the line CD is a parallel plane a distance $DF = \zeta$ above the fundamental plane. Let:





$K =$ The Sun's radius $= SA$

$k =$ The Moon's radius $= MB$

$G =$ The distance between the Sun and the Moon $= SM$

$f =$ The angle of the shadow cone $= EVF$

$c =$ The distance between the vertex of the cone and the fundamental plane $= VF$

$l =$ The radius of the shadow on the fundamental plane $= EF$

$L =$ The radius of the shadow on the parallel plane $= CD$

$\zeta =$ The distance from the parallel plane to the fundamental plane $= DF$

$z =$ The distance from the Moon to the fundamental plane $= MF$

We can see from the left diagram (the penumbral cone) that $SV + MV = G$. We can also see that $SA/SV = MB/MV = \sin(f)$. Thus:

$$\sin(f) = \frac{K+k}{G} \tag{9.11}$$

For the right diagram (the umbral cone), the vertex of the cone is below the parallel plane (total eclipse). Regardless, we can see that $SV - MV = G$ and thus:

$$\sin(f) = \frac{K-k}{G} \tag{9.12}$$

Equation 9.12 is true for annular eclipses as well (in which case the vertex of the cone is between the Moon and the parallel plane).
Then, because $MV = k/\sin(f)$, c is given by:

$$c = z \pm \frac{k}{\sin(f)} \tag{9.13}$$

the upper sign being used for the penumbra and the lower for the umbra. We then have:

$$\left.\begin{aligned} l &= c\tan(f) = z\tan(f) \pm k\sec(f) \\ L &= (c-\zeta)\tan(f) = l - \zeta\tan(f) \end{aligned}\right\} \tag{9.14}$$

For the umbral cone, $c - \zeta$ is negative when the vertex of the cone falls beneath the parallel plane, in which case we have total eclipse. Therefore we have L as a negative number when there is a total eclipse, and positive for partial and annular eclipses. l, being a distance, a positive quantity should be "correct", but keeping it as a negative number will be convenient later.

For brevity we will put:

$$\left.\begin{aligned} i &= \tan(f) \\ l &= ic \\ L &= l - i\zeta \end{aligned}\right\} \tag{9.15}$$

The quantities d, μ, x, y, i_1 (i for the penumbra), i_2 (i for the umbra), l_1 (l for the penumbra), l_2 (l for the umbra), and their derivatives are known as the **Besselian elements** of a solar eclipse.

Example 9.2 Determine the Besselian elements for the solar eclipse of April 8, 2024 for the time 18 : 00.

Solution
Let r, α, δ be the equatorial spherical coordinates of the Moon and r', α', δ' be the equatorial spherical coordinates of the Sun. From an ephemeris, we find that on April 8, 2024 at 18 : 00:

$$\begin{aligned} r &= 359\,780.727 \text{ km} \\ \alpha &= 1^h\, 9^m\, 4.27^s \\ \delta &= 7^\circ\, 41'\, 16.5'' \\ r' &= 149\,822\,802.516 \text{ km} \\ \alpha' &= 1^h\, 10^m\, 19.99^s \\ \delta' &= 7^\circ\, 27'\, 36.9'' \end{aligned}$$

Thus the cartesian coordinates of the Moon and Sun are:

$$\begin{aligned} v &= 340\,068.149 \text{ km} \\ u &= 107\,766.833 \text{ km} \\ w &= 46\,713.3280 \text{ km} \\ v' &= 141\,783\,968 \text{ km} \\ u' &= 44\,073\,369.0 \text{ km} \\ w' &= 20\,042\,873.5 \text{ km} \end{aligned}$$

(If vectors are already known, the above step may be skipped.)
Therefore, by equation 9.4:

$$\begin{aligned} G\cos(d)\cos(a) &= 141\,783\,968 - 340\,068.149 \ = 141\,273\,455 \text{ km} \\ G\cos(d)\sin(a) &= 44\,073\,369.0 - 107\,766.833 = 44\,771\,300.3 \text{ km} \\ G\sin(d) &= 20\,042\,873.5 - 46\,713.3280 = 19\,404\,611.7 \text{ km} \end{aligned}$$

Therefore:

$$\begin{aligned} G &= \sqrt{141\,273\,455^2 + 44\,771\,300.3^2 + 19\,404\,611.7^2} && = 149\,463\,030 \text{ km} \\ a &= \arctan(44\,771\,300.3, 141\,273\,455) && = 17^\circ\, 35'\, 2.58'' \\ d &= \arcsin(19\,404\,611.7/G) && = 7^\circ\, 27'\, 34.93'' \end{aligned}$$

Thus, by equation 9.5, the rotation matrix is:

$$R = \begin{bmatrix} -0.302104542 & 0.9532748008 & 0 \\ -0.12376256 & -0.039221882 & 0.991536420 \\ 0.945206683 & 0.299547656 & 0.12982884 \end{bmatrix}$$

Thus, by equation 9.6:

$$\begin{aligned} x &= -1968.0435 \text{ km} \\ y &= 1433.7449 \text{ km} \\ z &= 359\,772.487 \text{ km} \end{aligned}$$

Additionally, the sidereal time for $18:00$, April 8, 2024 is:

$$\Theta = 107^\circ\, 29'\, 7.05''$$

So, by equation 6.1, the hour angle of the shadow axis at the Greenwich (standard) meridian is:

$$\mu = 107^\circ\, 29'\, 7.05'' - 17^\circ\, 35'\, 2.58'' = 89^\circ\, 54'\, 4.47''$$

We find the remaining elements by equation 9.11, 9.12, 9.13, and 9.15 (and the radius data from example 9.1):

$$\begin{aligned} f_1 &= \arcsin\left(\frac{696\,000 + 1737.4}{149\,463\,030}\right) && = 16.80592'' \\ f_2 &= \arcsin\left(\frac{696\,000 - 1737.4}{149\,463\,030}\right) && = 16.72222'' \\ i_1 &= \tan(16.80592'') && = 0.0046683 \\ i_2 &= \tan(16.72222'') && = 0.0046451 \\ c_1 &= 359\,772.487 + \frac{1737.4}{\sin(16.80592'')} && = 731\,942.688 \text{ km} \\ c_2 &= 359\,772.487 - \frac{1737.4}{\sin(16.72222'')} && = -14\,260.536 \text{ km} \\ l_1 &= 0.0046683 \cdot 719\,445.246 && = 3416.961 \text{ km} \\ l_2 &= 0.0046451 \cdot -14\,260.536 && = -66.242 \text{ km} \end{aligned}$$

Expressing distances in terms of Earth equatorial radii ($R_E = 6378.137$ km) as is customary, the Besselian elements of this eclipse at 18 : 00 are:

$$\begin{aligned} d &= 7^\circ\ 27'\ 34.93'' \\ \mu &= 89^\circ\ 54'\ 4.47'' \\ x &= -0.30856088\ R_E \\ y &= 0.22479055\ R_E \\ i_1 &= 0.0046683 \\ i_2 &= 0.0046451 \\ l_1 &= 0.53573027\ R_E \\ l_2 &= -0.01038565\ R_E \end{aligned}$$

The quantity i will stay roughly constant for the whole eclipse for most reasonable cases. For the other derivatives, we take a time step of ± 15 minutes (thus $\Delta t = 0.5h$), and find:

Element	17 : 45	18 : 15
d	$7^\circ\ 27'\ 21.62''$	$7^\circ\ 27'\ 48.33''$
x	$-0.4364434\ R_E$	$-0.18070657\ R_E$
y	$0.15693548\ R_E$	$0.29258104\ R_E$
μ	$86^\circ\ 9'\ 0.73''$	$93^\circ\ 39'\ 8.07''$
l_1	$0.53571408\ R_E$	$0.53574486\ R_E$
l_2	$-0.01040175\ R_E$	$-0.01037113\ R_E$

Therefore:

$$\begin{aligned} d' &= \frac{7^\circ\ 27'\ 48.33'' - 7^\circ\ 27'\ 21.62''}{0.5h} \cdot \frac{\pi \text{ rad}}{180^\circ} &&= 0.00025898 \text{ rad}/h \\ x' &= \frac{-0.18070657 - (-0.4364434)}{0.5h} &&= 0.51147366\ R_E/h \\ y' &= \frac{0.29258104 - 0.15693548}{0.5h} &&= 0.27129112\ R_E/h \\ \mu' &= \frac{93^\circ\ 39'\ 8.07'' - 86^\circ\ 9'\ 0.73''}{0.5h} \cdot \frac{\pi \text{ rad}}{180^\circ} &&= 0.26187054 \text{ rad}/h \\ l_1' &= \frac{0.53574486 - 0.53571408}{0.5h} &&= 0.00006156\ R_E/h \\ l_2' &= \frac{-0.01037113 - (-0.01040175)}{0.5h} &&= 0.00006124\ R_E/h \end{aligned}$$

Taking a smaller time step for the derivative is better, but since we will be taking time steps of 15 minutes in later examples, we use 15 minutes here. See section 9.14 for a note on time steps.

9.5 The Outline of the Shadow on the Surface of the Earth

At a distance ζ above the fundamental plane, the shadow's radius is L. Therefore, for places on a parallel plane at the edge of the shadow, that is, for places where the eclipse is beginning or ending, the distance to the shadow must equal the radius of the shadow and therefore:

$$\Delta = L \tag{9.16}$$

Or, using equations 9.9 and 9.15:

$$(x - \xi)^2 + (y - \eta)^2 = (l - i\zeta)^2 \tag{9.17}$$

Which can also be expressed as (using equation 9.10):

$$\left.\begin{aligned}(l - i\zeta)\sin(Q) &= x - \xi \\ (l - i\zeta)\cos(Q) &= y - \eta\end{aligned}\right\} \quad (9.18)$$

Equation 9.18 is known as the *first fundamental equation of eclipse theory.*

To convert ξ, η, ζ to coordinates on the surface of the Earth, we have equations 9.7, restated here:

$$\left.\begin{aligned}\xi &= \rho\cos(\phi')\sin(\theta) \\ \eta &= \rho\sin(\phi')\cos(d) - \rho\cos(\phi')\sin(d)\cos(\theta) \\ \zeta &= \rho\sin(\phi')\sin(d) + \rho\cos(\phi')\cos(d)\cos(\theta)\end{aligned}\right\} \quad (9.7)$$

The five equations of 9.7 and 9.18 involve six variables, so one of them must be a free variable. Taking Q as a free variable and letting it range from 0° to 360° gives us the full outline of the shadow as Q describes the angle between the place of observation and the position of the shadow. Therefore we let Q range free.

However, we run into a problem: when using equations 9.7, we need ρ to find ϕ', but since ρ depends on ϕ', we cannot determine ρ until ϕ' is found. Friedrich Bessel found a clever workaround to this problem:
If ϕ is the geodetic latitude and distances are measured in units of Earth equatorial radii (such that $a = 1$), then we have equations 7.6:

$$\rho\cos(\phi') = \frac{\cos(\phi)}{\sqrt{1 - e^2\sin^2(\phi)}} \quad \text{and} \quad \rho\sin(\phi') = \frac{(1 - e^2)\sin(\phi)}{\sqrt{1 - e^2\sin^2(\phi)}}$$

Where e is the eccentricity of the Earth ellipsoid. If we define ϕ_1 such that:

$$\cos(\phi_1) = \rho\cos(\phi') = \frac{\cos(\phi)}{\sqrt{1 - e^2\sin^2(\phi)}}$$

we have:

$$\sin(\phi_1) = \sqrt{1 - \cos^2(\phi_1)} = \frac{\sin(\phi)\sqrt{1 - e^2}}{\sqrt{1 - e^2\sin^2(\phi)}}$$

or:

$$\left.\begin{aligned}\cos(\phi_1) &= \rho\cos(\phi') \\ \sqrt{1 - e^2}\sin(\phi_1) &= \rho\sin(\phi') \\ \therefore \tan(\phi) &= \frac{\tan(\phi_1)}{\sqrt{1 - e^2}}\end{aligned}\right\} \quad (9.19)$$

Now, equations 9.7 become:

$$\left.\begin{aligned}\xi &= \cos(\phi_1)\sin(\theta) \\ \eta &= \sin(\phi_1)\cos(d)\sqrt{1 - e^2} - \cos(\phi_1)\sin(d)\cos(\theta) \\ \zeta &= \sin(\phi_1)\sin(d)\sqrt{1 - e^2} + \cos(\phi_1)\cos(d)\cos(\theta)\end{aligned}\right\} \quad (9.20)$$

Let

$$\left.\begin{cases}\rho_1\sin(d_1) = \sin(d) \\ \rho_1\cos(d_1) = \cos(d)\sqrt{1 - e^2}\end{cases} \qquad \begin{cases}\rho_2\sin(d_2) = \sin(d)\sqrt{1 - e^2} \\ \rho_2\cos(d_2) = \cos(d)\end{cases}\right\} \quad (9.21)$$

Since ρ_1, d_1, ρ_2, and d_2 do not depend on Q, they can be calculated along with the Besselian elements. Note that the subscripts do not mean penumbra and umbra here: they are just there to distinguish

the variables.
Now, equations 9.20 become:

$$\left.\begin{aligned}\xi &= \cos(\phi_1)\sin(\theta)\\ \eta &= \rho_1\sin(\phi_1)\cos(d_1) - \rho_1\cos(\phi_1)\sin(d_1)\cos(\theta)\\ \zeta &= \rho_2\sin(\phi_1)\sin(d_2) + \rho_2\cos(\phi_1)\cos(d_2)\cos(\theta)\end{aligned}\right\}\quad(9.22)$$

If we put:

$$\eta_1 = \frac{\eta}{\rho_1} \quad (9.23)$$

And put ζ_1 such that:

$$\xi^2 + \eta_1^2 + \zeta_1^2 = 1 \quad (9.24)$$

Then:

$$\begin{aligned}\zeta_1^2 &= 1 - \xi^2 - \frac{\eta^2}{\rho_1^2}\\ &= 1 - \cos^2(\phi_1)\sin^2(\theta) - \sin^2(\phi_1)\cos^2(d_1) + 2\sin(\phi_1)\cos(d_1)\cos(\phi_1)\sin(d_1)\cos(\theta)\\ &\quad - \cos^2(\phi_1)\sin^2(d_1)\cos^2(\theta)\end{aligned}$$

Let us ignore the $2\sin(\phi_1)\cos(d_1)\cos(\phi_1)\sin(d_1)\cos(\theta)$ term for now. Carrying out algebra for the rest of the terms:

$$\begin{aligned}&1 - \cos^2(\phi_1)\sin^2(\theta) - \sin^2(\phi_1)\cos^2(d_1) - \cos^2(\phi_1)\sin^2(d_1)\cos^2(\theta)\\ =&1 - \cos^2(\phi_1)(1-\cos^2(\theta)) - \sin^2(\phi_1)(1-\sin^2(d_1)) - \cos^2(\phi_1)\sin^2(d_1)\cos^2(\theta)\\ =&1 - \cos^2(\phi_1) - \cos^2(\phi_1)\cos^2(\theta) - \sin^2(\phi_1) - \sin^2(\phi_1)\sin^2(d_1) - \cos^2(\phi_1)\sin^2(d_1)\cos^2(\theta)\\ =&-\sin^2(\phi_1)\sin^2(d_1) + \cos^2(\phi_1)\cos^2(\theta) - \cos^2(\phi_1)\sin^2(d_1)\cos^2(\theta)\\ =&-\sin^2(\phi_1)\sin^2(d_1) + \cos^2(\phi_1)\cos^2(\theta) - \cos^2(\phi_1)\cos^2(\theta)(1-\cos^2(d_1))\\ =&-\sin^2(\phi_1)\sin^2(d_1) + \cos^2(\phi_1)\cos^2(\theta) - \cos^2(\phi_1)\cos^2(\theta) + \cos^2(\phi_1)\cos^2(\theta)\cos^2(d_1)\\ =&-\sin^2(\phi_1)\sin^2(d_1) + \cos^2(\phi_1)\cos^2(\theta)\cos^2(d_1)\end{aligned}$$

When we add back the term we ignored, we find that the result is a perfect square.

$$\begin{aligned}\zeta_1^2 &= -\sin^2(\phi_1)\sin^2(d_1) + 2\sin(\phi_1)\sin(d_1)\cos(\phi_1)\cos(d_1)\cos(\theta) + \cos^2(\phi_1)\cos^2(d_1)\cos^2(\theta)\\ &= (\sin(\phi_1)\sin(d_1) + \cos(\phi_1)\cos(d_1)\cos(\theta))^2\end{aligned}$$

Thus:

$$\zeta_1 = \sin(\phi_1)\sin(d_1) + \cos(\phi_1)\cos(d_1)\cos(\theta)$$

Therefore, equations 9.22 can be further simplified to:

$$\left.\begin{aligned}\xi &= \cos(\phi_1)\sin(\theta)\\ \eta_1 &= \sin(\phi_1)\cos(d_1) - \cos(\phi_1)\sin(d_1)\cos(\theta)\\ \zeta_1 &= \sin(\phi_1)\sin(d_1) + \cos(\phi_1)\cos(d_1)\cos(\theta)\end{aligned}\right\}\quad(9.25)$$

If we multiply the second equation by $-\sin(d_1)$ and the third by $\cos(d_1)$ and sum them, we obtain:

$$\cos(\phi_1)\cos(\theta) = -\eta_1\sin(d_1) + \zeta_1\cos(d_1) \quad (9.26\text{-i})$$

Similarly, if we multiply the second equation by $\cos(d_1)$ and the third by $\sin(d_1)$ and sum them, we obtain:

$$\sin(\phi_1) = \eta_1\cos(d_1) + \zeta_1\sin(d_1) \quad (9.26\text{-ii})$$

Using these equations, we can find ζ if we know ζ_1. If we substitute equation 9.26 into the equation for ζ in equation 9.22 we get:

$$\begin{aligned}\zeta &= \rho_2(\eta_1\cos(d_1)+\zeta_1\sin(d_1))\sin(d_2)+\rho_2\cos(d_2)(-\eta_1\sin(d_1)+\zeta_1\cos(d_1))\\ &= \rho_2\eta_1\cos(d_1)\sin(d_2)+\rho_2\zeta_1\sin(d_1)\sin(d_2)-\rho_2\eta_1\cos(d_2)\sin(d_1)+\rho_2\zeta_1\cos(d_2)\cos(d_1)\\ &= -\rho_2\eta_1\sin(d_1-d_2)+\rho_2\zeta_1\cos(d_1-d_2)\end{aligned} \tag{9.27}$$

For planets with low flattening values, ζ_1 varies so little from ζ that we may substitute it for ζ in the equation for L. Therefore our problem now takes the following form. We first have equations 9.18 (with equation 9.23 substituted in) and 9.24:

$$\left.\begin{aligned}(l-i\zeta_1)\sin(Q) &= x-\xi\\ (l-i\zeta_1)\cos(Q) &= y-\rho_1\eta_1\\ \xi^2+\eta_1^2+\zeta_1^2 &= 1\end{aligned}\right\} \tag{9.28}$$

Which fully determine ξ, η_1 and ζ_1 for any value of Q. Now, from ξ, η_1 and ζ_1 we can use equations 9.25 and 9.26:

$$\left.\begin{aligned}\cos(\phi_1)\sin(\theta) &= \xi\\ \cos(\phi_1)\cos(\theta) &= -\eta_1\sin(d_1)+\zeta_1\cos(d_1)\\ \sin(\phi_1) &= \eta_1\cos(d_1)+\zeta_1\sin(d_1)\end{aligned}\right\} \tag{9.29}$$

Which fully determine ϕ_1 and θ. The actual geodetic latitude and longitude of the place are then given by equations 9.19 and 6.2:

$$\left.\begin{aligned}\tan(\phi) &= \frac{\tan(\phi_1)}{\sqrt{1-e^2}}\\ \lambda &= \theta-\mu\end{aligned}\right\} \tag{9.30}$$

In order to solve equation 9.28, let β and γ be defined such that:

$$\left.\begin{aligned}\sin(\beta)\sin(\gamma) &= x-l\sin(Q)\\ \sin(\beta)\cos(\gamma) &= \frac{y-l\cos(Q)}{\rho_1}\end{aligned}\right\} \tag{9.31}$$

Then, we have:

$$\left.\begin{aligned}\xi &= \sin(\beta)\sin(\gamma)+i\zeta_1\sin(Q)\\ \eta_1 &= \sin(\beta)\cos(\gamma)+\frac{i\zeta_1\cos(Q)}{\rho_1}\end{aligned}\right\} \tag{9.32}$$

Substituting these into the last equation of 9.28, we have:

$$(\sin(\beta)\sin(\gamma)+i\zeta_1\sin(Q))^2+\left(\sin(\beta)\cos(\gamma)+\frac{i\zeta_1\cos(Q)}{\rho_1}\right)^2+\zeta_1^2=1$$

$$\begin{aligned}\sin^2(\beta)\sin^2(\gamma)+2i\zeta_1\sin(\beta)\sin(\gamma)\sin(Q)+i^2\zeta_1^2\sin^2(Q)+\sin^2(\beta)\cos^2(\gamma)&\\ +\frac{2i\zeta_1\sin(\beta)\cos(\gamma)\cos(Q)}{\rho_1}+\frac{i^2\zeta_1^2\cos^2(Q)}{\rho_1^2}+\zeta_1^2&=1\end{aligned}$$

$$\begin{aligned}-\cos^2(\beta)+2i\zeta_1\sin(\beta)\sin(\gamma)\sin(Q)+i^2\zeta_1^2\sin^2(Q)&\\ +\frac{2i\zeta_1\sin(\beta)\cos(\gamma)\cos(Q)}{\rho_1}+\frac{i^2\zeta_1^2\cos^2(Q)}{\rho_1^2}+\zeta_1^2&=0\end{aligned}$$

Rearranging the terms, we get a quadratic in ζ_1:

$$\begin{aligned}&\left(1+i^2\sin^2(Q)+\frac{i^2\cos^2(Q)}{\rho_1^2}\right)\zeta_1^2\\ &+\left(2i\sin(\beta)\sin(\gamma)\sin(Q)+\frac{2i\sin(\beta)\cos(\gamma)\cos(Q)}{\rho_1}\right)\zeta_1-\cos^2(\beta)=0\end{aligned} \tag{9.33}$$

All the coefficients involve known numbers. If the approximation $\rho_1 = 1$ is taken, this equation simplifies down to:

$$(1 + i^2)\zeta_1^2 + 2i\sin(\beta)\cos(Q - \gamma)\zeta_1 - \cos^2(\beta) = 0 \tag{9.33*}$$

There are two solutions to these quadratics. To find the correct one, let us first say:

$Z =$ the zenith distance of the point Z (the point in the direction of the shadow axis).

Then, since Z points to right ascension a and declination d, the hour angle of this point is $\mu - a = \theta$, and thus, via the third equation of equation 6.3 (using the fact that zenith distance = 90°− altitude (equation 6.16)):

$$\cos(Z) = \sin(\phi)\sin(d) + \cos(\phi)\cos(d)\cos(\theta)$$

If we multiply both sides by $\sqrt{1 - e^2}/\left(\rho_1\sqrt{1 - e^2\sin^2(\phi)}\right)$, we see that we obtain the formula for ζ_1 in equation 9.25. Thus:

$$\cos(Z) = \zeta_1\rho_1\frac{\sqrt{1 - e^2\sin^2(\phi)}}{\sqrt{1 - e^2}} = \zeta_1\rho_1\frac{\sin(\phi)}{\sin(\phi_1)} \tag{9.34}$$

As $\rho_1\sin(\phi)/\sin(\phi_1)$ is a positive quantity, $\cos(Z)$ and ζ_1 have the same sign. But since we need the eclipse to be visible, Z cannot be more than 90° (or else the eclipse would be below the horizon) and thus $\cos(Z) > 0$ and therefore we can say that $\zeta_1 > 0$. The negative value would give the point on the other side of the Earth that would see the eclipse as well if the Earth were completely transparent.

Once ζ_1 has been found and if more accuracy is desired, we can correct for the $\zeta = \zeta_1$ approximation we made earlier. We substitute the value of ζ_1 just obtained into equation 9.27, obtain a value for ζ, then substitute that into the value for L to calculate new values for ξ, η_1 and ζ_1 using these equations:

$$\left.\begin{aligned} \xi &= x - (l - i\zeta)\sin(Q) \\ \eta_1\rho_1 &= y - (l - i\zeta)\cos(Q) \\ \zeta_1^2 &= 1 - \xi^2 - \eta_1^2 \end{aligned}\right\} \tag{9.28*}$$

Which should only involve simple substitution. The full procedure is shown in the following example.

Example 9.3 Find the outline of the Moon's penumbra at the time 18 : 00 during the eclipse of April 8, 2024.

Solution
The relevant Besselian elements (calculated in the previous example) are given here:

$$\begin{aligned} d &= 7^\circ\ 27'\ 34.93'' \\ \mu &= 89^\circ\ 54'\ 4.47'' \\ x &= -0.30856088\ R_E \\ y &= 0.22479055\ R_E \\ i_1 &= 0.0046683 \\ l_1 &= 0.53573027\ R_E \end{aligned}$$

As well as the eccentricity of the Earth spheroid:

$$e = 0.081819$$

Since ρ_1, d_1, ρ_2, and d_2 do not depend on Q, We first find ρ_1, d_1, ρ_2, and d_2 by equation 9.21:

$$\begin{aligned} \rho_1\sin(d_1) &= \sin(7^\circ\ 27'\ 34.93'') &&= 0.12982884 \\ \rho_1\cos(d_1) &= \cos(7^\circ\ 27'\ 34.93'')\sqrt{1 - 0.081819^2} &&= 0.98821200 \\ \therefore \rho_1 &= \sqrt{0.12982884^2 + 0.98821200^2} &&= 0.99670381 \\ \therefore d_1 &= \arctan(0.12982884, 0.98821200) &&= 7^\circ\ 29'\ 4.25'' \end{aligned}$$

$$
\begin{aligned}
\rho_2 \sin(d_2) &= \sin(7^\circ\, 27'\, 34.93'')\sqrt{1-0.081819^2} &&= 0.12939355\\
\rho_2 \cos(d_2) &= \cos(7^\circ\, 27'\, 34.93'') &&= 0.99153642\\
\therefore \rho_2 &= \sqrt{0.12939355^2 + 0.99153642^2} &&= 0.99994358\\
\therefore d_2 &= \arctan(0.12939355, 0.99153642) &&= 7^\circ\, 26'\, 5.89''
\end{aligned}
$$

Note that ρ_2 and d_2 are only needed for the optional correction step.

Now we choose a value for Q. I will give the procedure for $Q = 90^\circ$ in full detail.
We first find β and γ by equation 9.31:

$$
\begin{aligned}
\sin(\beta)\sin(\gamma) &= -0.30856088 - 0.53573027\sin(90^\circ) &&= -0.84429113\\
\sin(\beta)\cos(\gamma) &= \frac{0.22479055 - 0.53573027\cos(90^\circ)}{0.99670381} &&= 0.22553395\\
\therefore \gamma &= \arctan(-0.84429113, 0.22553395) &&= -75^\circ\, 2'\, 38.03''\\
\therefore \beta &= \arcsin(-0.84429113/\sin(-75^\circ\, 2'\, 38.03'')) &&= 60^\circ\, 54'\, 52.23''
\end{aligned}
$$

Now we have everything necessary to solve equation 9.33∗ (we use equation 9.33∗ because ρ_1 is very close to 1). Let us denote the first coefficient by c_1, the second by c_2, and the third by c_3:

$$
\begin{aligned}
c_1 &= 1 + 0.0046683^2 &&= 1.00002179\\
c_2 &= 2 \cdot 0.0046683 \sin(60^\circ\, 54'\, 52.23'')\cos(90^\circ - (-75^\circ\, 2'\, 38.03'')) &&= -0.00788288\\
c_3 &= -\cos^2(60^\circ\, 54'\, 52.23'') &&= 0.23630692
\end{aligned}
$$

Solving the quadratic and choosing the positive value for ζ_1 gives:

$$
\zeta_1 = \frac{-(-0.00788288) + \sqrt{(-0.00788288)^2 - 4 \cdot 1.00002179 \cdot 0.23630692}}{2 \cdot 1.00002179} = 0.49006614
$$

Now we find ξ and η_1 by equation 9.32:

$$
\begin{aligned}
\xi &= -0.84429113 + 0.0046683 \cdot 0.49006614 \sin(90^\circ) = -0.84200334\\
\eta_1 &= 0.22553395 + \frac{0.0046683 \cdot 0.49006614\cos(90^\circ)}{0.98821200} = 0.22553395
\end{aligned}
$$

Now for the optional correction step. We find ζ by equation 9.27:

$$
\begin{aligned}
\zeta &= -0.99994358 \cdot 0.22553395 \sin(7^\circ\, 29'\, 4.25'' - 7^\circ\, 26'\, 5.89'')\\
&\quad + 0.99994358 \cdot 0.49006614 \cos(7^\circ\, 29'\, 4.25'' - 7^\circ\, 26'\, 5.89'')\\
&= 0.48984331
\end{aligned}
$$

Then, we find the corrected values of ξ, η_1, and ζ_1 by equation 9.28∗:

$$
\begin{aligned}
\xi &= -0.30856088 - (0.53573027 - 0.0046683 \cdot 0.48984331)\sin(90^\circ) &&= -0.84200438\\
\eta_1 &= \frac{1}{0.99670381}(0.22479055 - (0.53573027 - 0.0046683 \cdot 0.48984331)\cos(90^\circ)) &&= 0.22553395\\
\therefore \zeta_1^2 &= 1 - (-0.84200438)^2 - (0.22553395)^2 &&= 0.24016307\\
\therefore \zeta_1 &= +\sqrt{0.24016307} &&= 0.49006435
\end{aligned}
$$

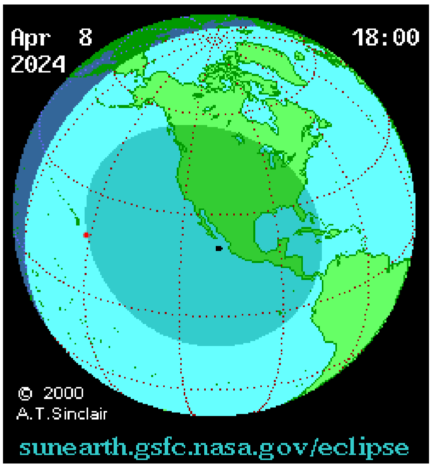
This correction to ξ, η_1, and ζ_1 is absolutely miniscule. The difference in the value for ξ obtained in this example is in the order of 0.0001%. Now, by equation 9.29:

$$
\begin{aligned}
\cos(\phi_1)\sin(\theta) &= -0.84200438\\
\cos(\phi_1)\cos(\theta) &= -0.22553395\sin(7^\circ\, 29'\, 4.25'') + 0.49006435\cos(7^\circ\, 29'\, 4.25'') &&= 0.45651141\\
\sin(\phi_1) &= 0.22553395\cos(7^\circ\, 29'\, 4.25'') + 0.49006435\sin(7^\circ\, 29'\, 4.25'') &&= 0.28744732\\
\therefore \phi_1 &= \arcsin(0.28744732) &&= 16^\circ\, 42'\, 18.69''\\
\therefore \theta &= \arctan(-0.84200438, 0.45651141) &&= -61^\circ\, 32'\, 4.85''
\end{aligned}
$$

Finally, the latitude and longitude of the place are given by equation 9.30:

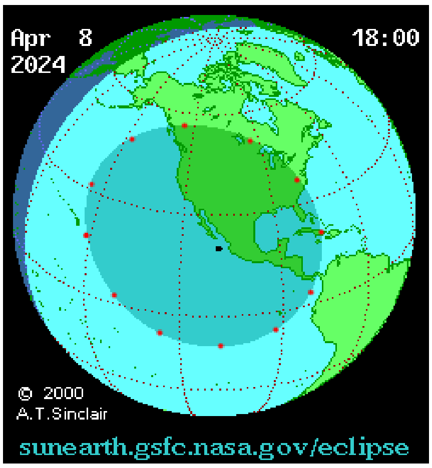
$$\begin{aligned}
\tan(\phi) &= \frac{\tan(16^\circ\ 42'\ 18.69'')}{\sqrt{1-0.081819^2}} &&= 0.30112276 \\
\therefore \phi &= \arctan(0.30112276) &&= 16^\circ\ 45'\ 29.68'' \\
\lambda &= -61^\circ\ 32'\ 4.85'' - 89^\circ\ 54'\ 4.47'' &&= 208^\circ\ 33'\ 50.68''
\end{aligned}$$

The point we just calculated is the highlighted red point on this map:



By ranging Q from 0° to 360° we can get the shape of the full shadow:

Q	ϕ	λ
0°	−10° 52′	251° 47′
30°	−7° 39′	234° 40′
60°	2° 23′	219° 41′
90°	16° 45′	208° 34′
120°	32° 53′	203° 36′
150°	47° 53′	211° 10′
180°	56° 1′	236° 40′
210°	51° 3′	266° 14′
240°	36° 57′	281° 3′
270°	20° 23′	283° 50′
300°	5° 4′	278° 50′
330°	−6° 12′	267° 37′



If solutions do not exist, that means that that part of the shadow falls off the surface of the Earth.

9.6 Beginning / Ending Condition

We have found the locations on the edge of the shadow but not if the eclipse is about to begin or about to end in those places. If at time t a point is on the edge of the shadow, then whether the point is inside or outside the shadow at the next instant time $t + dt$ will determine if the eclipse was beginning or ending at that place. This can be expressed in terms of equation 9.16: the eclipse is beginning or ending depending on Δ is becoming greater or less than L at time $t + dt$. We can express this as the derivative of the function $\Delta^2 - L^2$: *if the derivative is positive then the distance is increasing and the eclipse is ending. If the derivative is negative then the distance is decreasing and the eclipse is beginning.*

It is very useful now to define some vectors. Let the vector $\boldsymbol{\varrho}$ be defined as the position of the observer in the fundamental frame:

$$\boldsymbol{\varrho} = \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \tag{9.35}$$

And $\mathbf{r}$ be defined as the position of the Moon's shadow at the level of the observer:

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ \zeta \end{bmatrix} \tag{9.36}$$

Now the vector pointing from the shadow to the observer $\boldsymbol{\Delta}$ can be defined as:

$$\boldsymbol{\Delta} = \mathbf{r} - \boldsymbol{\varrho} = \begin{bmatrix} x - \xi \\ y - \eta \\ 0 \end{bmatrix} \tag{9.37}$$

The magnitude of $\boldsymbol{\Delta}$ is Δ (equation 9.9).

Now, $\Delta^2 - L^2$ becomes:

$$\boldsymbol{\Delta} \cdot \boldsymbol{\Delta} - L^2$$

Taking the derivative (which we will denote by P) with respect to time, we get:

$$P = 2\boldsymbol{\Delta} \cdot \boldsymbol{\Delta}' - 2LL'$$

Which, because $|\boldsymbol{\Delta}| = L$ at the instant t, we can reduce to:

$$P = L(\hat{\boldsymbol{\Delta}} \cdot \boldsymbol{\Delta}' - L')$$

Where $\hat{\boldsymbol{\Delta}}$ is the unit vector in the direction of $\boldsymbol{\Delta}$. The 2 was omitted since only the sign of the derivative is important. If we now set:

$$P' = \hat{\boldsymbol{\Delta}} \cdot \boldsymbol{\Delta}' - L'$$

Then

$$P = LP' \tag{9.38}$$

And so can see that *if P' and L have like signs, then P is positive and the eclipse is ending. If P' and L have opposite signs, then P is negative and the eclipse is beginning.* Let us now develop the quantity P'.

We can safely say that i is constant and therefore:

$$L' = l' - i\zeta'$$

Also, By equation 9.37,

$$\boldsymbol{\Delta}' = \mathbf{r}' - \boldsymbol{\varrho}'$$

Therefore:

$$P' = \hat{\mathbf{\Delta}} \cdot \mathbf{r}' - \hat{\mathbf{\Delta}} \cdot \boldsymbol{\varrho}' - (l' - i\zeta') \tag{9.39}$$

So we need to develop the quantities in the right hand side.

Let us first calculate $\boldsymbol{\varrho}'$, which involves ξ', η', and ζ'. If the observer is at geocentric equatorial coordinates

$$\begin{bmatrix} \rho\cos(\phi')\cos(\Theta_L) \\ \rho\cos(\phi')\sin(\Theta_L) \\ \rho\sin(\phi') \end{bmatrix}$$

Then they are rotating about the z-axis as Θ_L changes. Thus the rotational velocity vector in equatorial coordinates is:

$$\begin{bmatrix} 0 \\ 0 \\ d\Theta/dt \end{bmatrix}$$

Because $d\Theta_L/dt = d\Theta/dt + d/dt(l) = d\Theta/dt$. But we want the velocity vector relative to the shadow axis. The shadow axis (in equatorial coordinates) is also rotating about the x and z axes:

$$\begin{bmatrix} -dd/dt \\ 0 \\ da/dt \end{bmatrix}$$

See equation 9.5 for rationale of the signs. Now we can subtract the two vectors and obtain for the relative rotation vector (in equatorial coordinates):

$$\boldsymbol{\omega}_E = \begin{bmatrix} dd/dt \\ 0 \\ d\Theta/dt - da/dt \end{bmatrix} = \begin{bmatrix} d' \\ 0 \\ \mu' \end{bmatrix}$$

To transform into fundamental coordinates, we only need to multiply by a rotation about the equatorial x axis by $90° - d$ since the transformation about the z axis by $a - 90°$ does not change anything about the rotation vector. Therefore:

$$\boldsymbol{\omega} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(90° - d) & \sin(90° - d) \\ 0 & -\sin(90° - d) & \cos(90° - d) \end{bmatrix} \begin{bmatrix} d' \\ 0 \\ \mu' \end{bmatrix} = \begin{bmatrix} d' \\ \cos(d)\mu' \\ \sin(d)\mu' \end{bmatrix}$$

The cartesian velocity vector in the fundamental frame is then the cross product of the fundamental rotation vector and the fundamental position vector:

$$\boldsymbol{\varrho}' = \begin{bmatrix} \xi' \\ \eta' \\ \zeta' \end{bmatrix} = \boldsymbol{\omega} \times \boldsymbol{\varrho} = \begin{bmatrix} d' \\ \cos(d)\mu' \\ \sin(d)\mu' \end{bmatrix} \times \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = \begin{bmatrix} \mu'(-\eta\sin(d) + \zeta\cos(d)) \\ \mu'\xi\sin(d) - d'\zeta \\ -\mu'\xi\cos(d) + d'\eta \end{bmatrix} \tag{9.40}$$

We can substitute $\boldsymbol{\varrho}' = \boldsymbol{\omega} \times \boldsymbol{\varrho}$ into equation 9.39:

$$P' = \hat{\mathbf{\Delta}} \cdot \mathbf{r}' - \hat{\mathbf{\Delta}} \cdot (\boldsymbol{\omega} \times \boldsymbol{\varrho}) - (l' - i\zeta')$$

Using equation 9.37 we can write:

$$P' = \hat{\mathbf{\Delta}} \cdot \mathbf{r}' - \hat{\mathbf{\Delta}} \cdot (\boldsymbol{\omega} \times (\mathbf{r} - \mathbf{\Delta})) - (l' - i\zeta') \tag{9.41}$$

The second term $\hat{\mathbf{\Delta}} \cdot (\boldsymbol{\omega} \times (\mathbf{r} - \mathbf{\Delta}))$ can be expanded as:

$$\begin{aligned} \hat{\mathbf{\Delta}} \cdot (\boldsymbol{\omega} \times (\mathbf{r} - \mathbf{\Delta})) &= \hat{\mathbf{\Delta}} \cdot (\boldsymbol{\omega} \times \mathbf{r} - \boldsymbol{\omega} \times \mathbf{\Delta}) \\ &= \hat{\mathbf{\Delta}} \cdot (\boldsymbol{\omega} \times \mathbf{r}) - \hat{\mathbf{\Delta}} \cdot (\boldsymbol{\omega} \times \mathbf{\Delta}) \end{aligned}$$

The second term, the scalar triple product $\hat{\boldsymbol{\Delta}} \cdot (\boldsymbol{\omega} \times \boldsymbol{\Delta})$, is:

$$\hat{\boldsymbol{\Delta}} \cdot (\boldsymbol{\omega} \times \boldsymbol{\Delta}) = \boldsymbol{\omega} \cdot (\boldsymbol{\Delta} \times \hat{\boldsymbol{\Delta}}) = \boldsymbol{\omega} \cdot \mathbf{0} = 0$$

Thus,

$$\hat{\boldsymbol{\Delta}} \cdot (\boldsymbol{\omega} \times (\mathbf{r} - \boldsymbol{\Delta})) = \hat{\boldsymbol{\Delta}} \cdot (\boldsymbol{\omega} \times \mathbf{r}) \tag{9.42}$$

$\boldsymbol{\omega} \times \mathbf{r}$ is:

$$\boldsymbol{\omega} \times \mathbf{r} = \begin{bmatrix} d' \\ \cos(d)\mu' \\ \sin(d)\mu' \end{bmatrix} \times \begin{bmatrix} x \\ y \\ \zeta \end{bmatrix} = \begin{bmatrix} \mu'(-y\sin(d) + \zeta\cos(d)) \\ \mu' x \sin(d) - d'\zeta \\ -\mu' x\cos(d) + d'y \end{bmatrix} = \mu' \begin{bmatrix} -y\sin(d) \\ x\sin(d) \\ -x\cos(d) \end{bmatrix} - \zeta \begin{bmatrix} -\mu'\cos(d) \\ d' \\ -d'y/\zeta \end{bmatrix} \tag{9.43}$$

For the third term of equation 9.41, notice that in equation 9.40 we can write ζ' as a vector dot product:

$$\zeta' = -\mu'\xi\cos(d) + d'\eta = \begin{bmatrix} -\mu'\cos(d) \\ d' \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = \begin{bmatrix} -\mu'\cos(d) \\ d' \\ 0 \end{bmatrix} \cdot \boldsymbol{\varrho} = \begin{bmatrix} -\mu'\cos(d) \\ d' \\ 0 \end{bmatrix} \cdot (\mathbf{r} - \boldsymbol{\Delta}) \tag{9.44}$$

To finish, we have from equation 9.10 and 9.16:

$$\boldsymbol{\Delta} = (l - i\zeta)\hat{\boldsymbol{\Delta}} = (l - i\zeta) \begin{bmatrix} \sin(Q) \\ \cos(Q) \\ 0 \end{bmatrix} \tag{9.45}$$

Substituting equations equation 9.45 into equation 9.44 and 9.42, 9.43, and 9.44 into equation 9.41, we have:

$$P' = \hat{\boldsymbol{\Delta}} \cdot \left(\mathbf{r}' - \mu' \begin{bmatrix} -y\sin(d) \\ x\sin(d) \\ -x\cos(d) \end{bmatrix} + \zeta \begin{bmatrix} -\mu'\cos(d) \\ d' \\ -d'y/\zeta \end{bmatrix} \right) - l' + i \begin{bmatrix} -\mu'\cos(d) \\ d' \\ 0 \end{bmatrix} \cdot (\mathbf{r} - (l - i\zeta)\hat{\boldsymbol{\Delta}})$$

$$\begin{aligned} &= \hat{\boldsymbol{\Delta}} \cdot \mathbf{r}' - \mu'\hat{\boldsymbol{\Delta}} \cdot \begin{bmatrix} -y\sin(d) \\ x\sin(d) \\ -x\cos(d) \end{bmatrix} + \zeta\hat{\boldsymbol{\Delta}} \cdot \begin{bmatrix} -\mu'\cos(d) \\ d' \\ -d'y/\zeta \end{bmatrix} - l' + i \begin{bmatrix} -\mu'\cos(d) \\ d' \\ 0 \end{bmatrix} \cdot \mathbf{r} \\ &\quad - li \begin{bmatrix} -\mu'\cos(d) \\ d' \\ 0 \end{bmatrix} \cdot \hat{\boldsymbol{\Delta}} + i^2\zeta \begin{bmatrix} -\mu'\cos(d) \\ d' \\ 0 \end{bmatrix} \cdot \hat{\boldsymbol{\Delta}} \end{aligned}$$

If we collect the terms as $\hat{\boldsymbol{\Delta}}\cdot$ and $\zeta\hat{\boldsymbol{\Delta}}\cdot$, we get:

$$\begin{aligned} P' &= \hat{\boldsymbol{\Delta}} \cdot \left(\mathbf{r}' - \mu' \begin{bmatrix} -y\sin(d) \\ x\sin(d) \\ -x\cos(d) \end{bmatrix} - li \begin{bmatrix} -\mu'\cos(d) \\ d' \\ 0 \end{bmatrix} \right) \\ &\quad + \zeta\hat{\boldsymbol{\Delta}} \cdot \left(\begin{bmatrix} -\mu'\cos(d) \\ d' \\ -d'y/\zeta \end{bmatrix} + i^2 \begin{bmatrix} -\mu'\cos(d) \\ d' \\ 0 \end{bmatrix} \right) - l' + i \begin{bmatrix} -\mu'\cos(d) \\ d' \\ 0 \end{bmatrix} \cdot \mathbf{r} \end{aligned} \tag{9.46}$$

Because the third element of $\hat{\boldsymbol{\Delta}}$ is 0 (see equations 9.37 and 9.45),

$$\hat{\boldsymbol{\Delta}} \cdot \begin{bmatrix} -\mu'\cos(d) \\ d' \\ -d'y/\zeta \end{bmatrix} = \hat{\boldsymbol{\Delta}} \cdot \begin{bmatrix} -\mu'\cos(d) \\ d' \\ 0 \end{bmatrix}$$

And therefore we can write equation 9.46 as:

$$P' = \hat{\boldsymbol{\Delta}} \cdot \left(\mathbf{r}' - \mu' \begin{bmatrix} -y\sin(d) \\ x\sin(d) \\ -x\cos(d) \end{bmatrix} - li \begin{bmatrix} -\mu'\cos(d) \\ d' \\ 0 \end{bmatrix} \right) + \zeta(i^2+1)\hat{\boldsymbol{\Delta}} \cdot \begin{bmatrix} -\mu'\cos(d) \\ d' \\ 0 \end{bmatrix} - l' + i \begin{bmatrix} -\mu'\cos(d) \\ d' \\ 0 \end{bmatrix} \cdot \mathbf{r}$$

Which, when all the vectors are substituted in, becomes:

$$\begin{aligned} P' = &\begin{bmatrix}\sin(Q)\\ \cos(Q)\\ 0\end{bmatrix} \cdot \left(\begin{bmatrix}x'\\ y'\\ \zeta'\end{bmatrix} - \mu' \begin{bmatrix}-y\sin(d)\\ x\sin(d)\\ -x\cos(d)\end{bmatrix} - li \begin{bmatrix}-\mu'\cos(d)\\ d'\\ 0\end{bmatrix} \right) \\ &+ \zeta(i^2+1) \begin{bmatrix}\sin(Q)\\ \cos(Q)\\ 0\end{bmatrix} \cdot \begin{bmatrix}-\mu'\cos(d)\\ d'\\ 0\end{bmatrix} - l' + i \begin{bmatrix}-\mu'\cos(d)\\ d'\\ 0\end{bmatrix} \cdot \begin{bmatrix}x\\ y\\ \zeta\end{bmatrix} \end{aligned} \tag{9.47}$$

Even though P' now involves all known variables, for the sake of brevity it is conventional to define three new variables a', b', and c' such that P' can be written as:

$$P' = \begin{bmatrix}\sin(Q)\\ \cos(Q)\\ 0\end{bmatrix} \cdot \begin{bmatrix}c'\\ -b'\\ 0\end{bmatrix} + \zeta(i^2+1) \begin{bmatrix}\sin(Q)\\ \cos(Q)\\ 0\end{bmatrix} \cdot \begin{bmatrix}-\mu'\cos(d)\\ d'\\ 0\end{bmatrix} + a' \tag{9.48}$$

Matching equation 9.48 to equation 9.47, we get for a', b', and c':

$$\left.\begin{aligned} a' &= -l' + \mu' ix\cos(d) + yid' \\ b' &= -y' + \mu' x\sin(d) + lid' \\ c' &= x' + \mu' y\sin(d) + li\mu'\cos(d) \end{aligned}\right\} \tag{9.49}$$

These variables do not depend on Q and therefore can be calculated along with the Besselian elements. Now, expanding out equation 9.48 we get:

$$P' = c'\sin(Q) - b'\cos(Q) - \zeta(i^2+1)\mu'\cos(d)\sin(Q) + \zeta(i^2+1)d'\cos(Q) + a'$$

Collecting terms as $\sin(Q)$ and $\cos(Q)$, we obtain:

$$P' = a' + (c' - (i^2+1)\zeta\mu'\cos(d))\sin(Q) + (-b' + (i^2+1)\zeta d')\cos(Q) \tag{9.50}$$

Equation 9.50 is known as the *second fundamental equation of eclipse theory.* Note that the approximations $id' = 0$ and $i^2 + 1 = 1$ may be made here without much loss of precision (in most reasonable cases), which gives:

$$P' = a' + (c' - \zeta\mu'\cos(d))\sin(Q) + (-b' + \zeta d')\cos(Q) \tag{9.50*}$$

With

$$\left.\begin{aligned} a' &= -l' + \mu' ix\cos(d) \\ b' &= -y' + \mu' x\sin(d) \\ c' &= x' + \mu' y\sin(d) + li\mu'\cos(d) \end{aligned}\right\} \tag{9.49*}$$

Also, ζ can also be substituted for ζ_1 in either equation 9.50 or 9.50$*$ if wanted.

Notice that if $P' = 0$, then by equation 9.38, $P = 0$ and therefore the eclipse is *neither beginning nor ending* (or beginning and ending at the same time if you prefer). This means that if $P' = 0$ at a certain time and location, the eclipse there is at maximum: at that time the Sun is obscured by the Moon as much as it ever will be at that location. This will be important later.

Example 9.4 Determine whether the eclipse was beginning or ending at the point for $Q = 90°$ determined in the last example.

Solution

The relevant Besselian elements are:

$$\begin{aligned}
d &= 7° \, 27' \, 34.93'' \\
x &= -0.30856088 \, R_E \\
y &= 0.22479055 \, R_E \\
i_1 &= 0.0046683 \\
l_1 &= 0.53573027 \, R_E \\
d' &= 0.00025898 \text{ rad}/h \\
x' &= 0.51147366 \, R_E/h \\
y' &= 0.27129112 \, R_E/h \\
\mu' &= 0.26187054 \text{ rad}/h \\
l_1' &= 0.00006156 \, R_E/h
\end{aligned}$$

Also, ζ for $Q = 90°$ was:

$$\zeta = 0.48984331$$

We first find a', b', and c' by equation 9.49:

$$\begin{aligned}
a' &= -0.00006156 + 0.26187054 \cdot 0.0046683 \cdot (-0.30856088) \cos(7° \, 27' \, 34.93'') \\
&\quad + 0.22479055 \cdot 0.0046683 \cdot 0.00025898 \\
&= -0.000435308 \\
b' &= -0.27129112 + 0.26187054 \cdot (-0.30856088) \sin(7° \, 27' \, 34.93'') \\
&\quad + 0.53573027 \cdot 0.0046683 \cdot 0.00025898 \\
&= -0.28178103 \\
c' &= 0.51147366 + 0.26187054 \cdot 0.22479055 \sin(7° \, 27' \, 34.93'') \\
&\quad + 0.53573027 \cdot 0.0046683 \cdot 0.26187054 \cos(7° \, 27' \, 34.93'') \\
&= 0.51976555
\end{aligned}$$

Then we find P' by equation 9.50:

$$\begin{aligned}
P' &= -0.000435308 + (0.51976555 - (0.0046683^2 + 1) \cdot 0.48984331 \cdot 0.26187054 \cos(7° \, 27' \, 34.93'')) \sin(90°) \\
&\quad + (-(-0.28178103) + (0.0046683^2 + 1) \cdot 0.48984331 \cdot 0.00025898) \cos(90°) \\
&= 0.39213761 > 0
\end{aligned}$$

Since $P' > 0$ and $L > 0$ (penumbra), $P > 0$ by equation 9.38 and therefore the eclipse was ending.

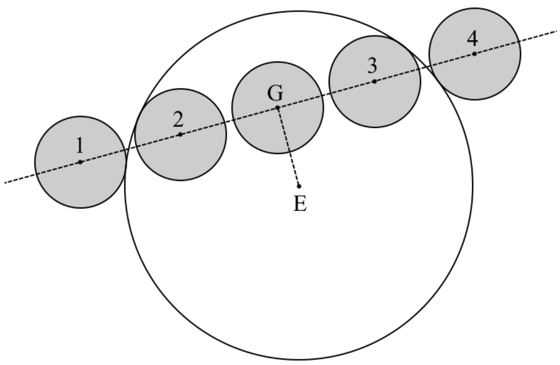
For comparison, equation 9.49$*$ gives:

$$\begin{aligned}
a' &= -0.00006156 + 0.26187054 \cdot 0.0046683 \cdot (-0.30856088) \cos(7° \, 27' \, 34.93'') \\
&= -0.000435580 \\
b' &= -0.27129112 + 0.26187054 \cdot (-0.30856088) \sin(7° \, 27' \, 34.93'') \\
&= -0.28178168 \\
c' &= 0.51147366 + 0.26187054 \cdot 0.22479055 \sin(7° \, 27' \, 34.93'') \\
&\quad + 0.53573027 \cdot 0.0046683 \cdot 0.26187054 \cos(7° \, 27' \, 34.93'') \\
&= 0.51976555
\end{aligned}$$

And equation 9.50* gives:

$$\begin{aligned} P' &= -0.000435580 + (0.51976555 - 0.48984331 \cdot 0.26187054 \cos(7^\circ\, 27'\, 34.93'')) \sin(90^\circ) \\ &\quad + (-(-0.28178168) + 0.48984331 \cdot 0.00025898) \cos(90^\circ) \\ &= 0.39214011 > 0 \end{aligned}$$

9.7 Contacts



As the shadow of the Moon (the shaded circle) passes over the Earth (the larger circle centered on E), there are a few key times of interest.

The times and points at which the shadow cone of the Moon is tangent to the Earth are known as *contacts*. There are four contacts, labeled 1, 2, 3, and 4 in the diagram:

1. First External Contact (External Ingress)
2. First Internal Contact (Internal Ingress)
3. Last Internal Contact (Internal Egress)
4. . Last External Contact (External Egress)

The point labeled G is called "Greatest Eclipse" and will be discussed later.

If the shadow in question is the penumbra, the contact points are called P_1, P_2, P_3, and P_4 respectively, and if the shadow is the umbra, they are called U_1, U_2, U_3, and U_4 respectively.

Evidently, P_1 and P_4 are the times at which the eclipse begins and ends globally, and U_1 and U_2 are the times at which the total (or annular) eclipse begins and ends globally.

Also, notice that because the shadow cone is tangent to the Earth, at the locations of contact the eclipse must be at the local horizon. For our purposes, this means that the zenith distance of the point Z (the point that points in the direction of the shadow axis) is 90°, and therefore by equation 9.34, $\zeta_1 = 0$. This means that, by equation 9.24, for contacts:

$$\xi^2 + \eta_1^2 = 1 \tag{9.51}$$

On the fundamental plane, let's define m, M, p, and γ such that:

$$\left.\begin{cases} m\sin(M) = x \\ m\cos(M) = y \end{cases} \qquad \begin{cases} p\sin(\gamma) = \xi \\ p\cos(\gamma) = \eta \end{cases}\right\} \tag{9.52}$$

Then, the distance from the center of the Earth to (x, y) is m and the distance to (ξ, η) is p.
Thus, at the contacts we have:

$$m = p \pm l$$

Where the top sign is for external contacts and the bottom is for internal contacts, and we use the *absolute value* of l (because it is used as a distance here, the umbral shadow must also be written positive here). Therefore, at contacts, we have:

$$\left.\begin{aligned}(p \pm l)\sin(M) &= x\\ (p \pm l)\cos(M) &= y\end{aligned}\right\} \quad (9.53)$$

For reasonable cases, x and y are very close to linear. Thus, if we say that at time T_0 the coordinates of the shadow of the Moon were x_0 and y_0 and their derivatives were x' and y', we have at time τ after T_0:

$$\begin{aligned}x &= x_0 + x'\tau\\ y &= y_0 + y'\tau\end{aligned}$$

If we now put:

$$\left.\begin{cases}m_0\sin(M_0) = x_0\\ m_0\cos(M_0) = y_0\end{cases} \qquad \begin{cases}n\sin(N) = x'\\ n\cos(N) = y'\end{cases}\right\} \quad (9.54)$$

Then equations 9.53 become:

$$\begin{aligned}(p \pm l)\sin(M) &= m_0\sin(M_0) + \tau n\sin(N)\\ (p \pm l)\cos(M) &= m_0\cos(M_0) + \tau n\cos(N)\end{aligned}$$

If we subtract N from all angles in the above equation, we get:

$$\begin{aligned}(p \pm l)\sin(M - N) &= m_0\sin(M_0 - N)\\ (p \pm l)\cos(M - N) &= m_0\cos(M_0 - N) + \tau n\end{aligned}$$

Now, if we put $\psi = M - N$, we have, for the time of contacts:

$$\left.\begin{aligned}\sin(\psi) &= \frac{m_0\sin(M_0 - N)}{p \pm l}\\ \tau &= \frac{(p \pm l)\cos(\psi) - m_0\cos(M_0 - N)}{n}\\ T &= T_0 + \tau\end{aligned}\right\} \quad (9.55)$$

The first equation has two solutions: we take $\psi = \arcsin()$ for last contacts and $\psi = 180° - \arcsin()$ for first contacts. Notice that in the first equation, for interior contacts, if $p-l$ is less than $m_0\sin(M_0-N)$, then $\sin(\psi)$ exceeds 1 and therefore no ψ exists: this means that there are no interior contacts and there is always a part of the shadow that misses the Earth.

In these formulae, we have everything we need except p. Because (ξ, η) must be on the edge of the Earth, p must be very close to 1 and we can start with $p = 1$ as a first approximation. Once getting the first approximation for ψ, we have $M = N + \psi$, but also since at the contacts (x, y) and (ξ, η) must be on the same line (see diagram), $M = \gamma$ and thus:

$$\gamma = N + \psi \quad (9.56)$$

Now, if we put $\xi = \sin(\gamma')$, then by equation 9.51, $\eta_1 = \cos(\gamma')$, and then because $\eta = \rho_1\eta_1$ (equation 9.23), we have:

$$\left.\begin{aligned}p\sin(\gamma) &= \sin(\gamma')\\ p\cos(\gamma) &= \rho_1\cos(\gamma')\end{aligned}\right\} \quad (9.57)$$

From these we can deduce:

$$\left.\begin{aligned}\gamma' &= \arctan(\rho_1\sin(\gamma), \cos(\gamma))\\ p &= \frac{\sin(\gamma')}{\sin(\gamma)} = \frac{\rho_1\cos(\gamma')}{\cos(\gamma)}\end{aligned}\right\} \quad (9.58)$$

Which yields a second approximation for p which yields another value of ψ. As many repetitions can be taken until convergence is reached. After repetition, a value for γ' gives:

$$\left.\begin{aligned} \xi &= \sin(\gamma') \\ \eta_1 &= \cos(\gamma') \\ \zeta_1 &= 0 \end{aligned}\right\} \quad (9.59)$$

From which equations 9.29 and 9.30 give the location of contact on the surface of the Earth.

Example 9.5 Find the time and location of the P_1, P_2, P_3, and P_4 contacts of the solar eclipse of April 8, 2024.

Solution

Elements will be taken from previous examples and all angles will be given in radians. We will take $T_0 = 18:00$. Equations 9.54 give:

$$\begin{aligned} m_0 \sin(M_0) &= -0.30856086 \\ m_0 \cos(M_0) &= 0.22479055 \\ \therefore m_0 &= \sqrt{(-0.30856086)^2 + 0.22479055^2} = 0.38175987 \\ \therefore M_0 &= \arctan(-0.30856086, 0.22479055) = -0.94118943 \text{ rad} \\ n \sin(N) &= 0.51147366 \\ n \cos(N) &= 0.27129112 \\ \therefore n &= \sqrt{0.51147366^2 + 0.27129112} \qquad = 0.57896820 \\ \therefore N &= \arctan(0.51147366, 0.27129112) \quad = 1.08311714 \text{ rad} \end{aligned}$$

Now, taking $p = 1$ as a first approximation, for the exterior contacts, equations 9.55 give:

$$\begin{aligned} \sin(\psi) &= \frac{0.38175987 \sin(-0.94118943 - 1.08311714)}{1 + 0.53573027} \\ &= -0.22345693 \\ \psi_{\text{First External}} &= \pi - \arcsin(-0.22345693) \\ &= 3.3669523 \text{ rad} \\ \tau_{\text{First External}} &= \frac{-(1 + 0.53573027)\cos(-0.22535964) - 0.38175987\cos(-0.94118943 - 1.08311714)}{0.57896820} \\ &= -2.29656737h \\ T_{\text{First External}} &= 18:00 - 2.29656737h = 15:42:12 \end{aligned}$$

$$\begin{aligned} \psi_{\text{Last External}} &= \arcsin(-0.22345693) \\ &= -0.22535964 \text{ rad} \\ \tau_{\text{Last External}} &= \frac{(1 + 0.53573027)\cos(-0.22535964) - 0.38175987\cos(-0.94118943 - 1.08311714)}{0.57896820} \\ &= 2.87434701h \\ T_{\text{Last External}} &= 18:00 + 2.87434701h = 20:52:28 \end{aligned}$$

Then by equation 9.56 and 9.58: (For the first external contact and using ρ_1 for $18:00$)

$$\begin{aligned} \gamma &= 1.08311714 + 3.3669523 && = 4.45006944 \\ \gamma' &= \arctan(0.99670381\sin(4.45006944), \cos(4.45006944)) && = -1.83394395 \\ p &= \frac{\sin(-1.83394395)}{\sin(4.45006944)} && = 0.999777311 \end{aligned}$$

Similarly, for the last external contact:

$$\gamma = 1.08311714 + (-0.22535964) = 0.8577575$$
$$\gamma' = \arctan(0.99670381\sin(0.8577575), \cos(0.8577575)) = 0.85612355$$
$$p = \frac{\sin(0.85612355)}{\sin(0.85612355)} = 0.99858559$$

Repeating this computation but using $p - l$ in equation 9.55 gives the times for internal contact:

	P_2	P_3
m_0	0.38175987	0.38175987
M_0	−0.94118943	−0.94118943
n	0.57896820	0.57896820
N	1.08311714	1.08311714
$\sin(\psi)$	−0.7391599	−0.7391599
ψ	3.97341485	−0.83182219
τ	−0.25120821	0.82898786
$T_0 + \tau = T$	17 : 44 : 56	18 : 49 : 44
$N + \psi = \gamma$	5.05653199	0.25129495
γ'	−1.22560336	0.25050091
p	0.99962314	0.99690667

Rounding the contact times to the nearest minute the Besselian elements are (using the method of example 9.2): (For a more accurate calculation, calculate Besselian elements at exactly those times we calculated. We only round here because the ephemeris we are using only gives minute precision.)

Element	15 : 42 : 00	20 : 52 : 00	17 : 45 : 00	18 : 50 : 00
d	0.12959946	0.13094043	0.13013186	0.13041266
μ	0.9667687	2.31976942	1.50360469	1.78729814
ρ_1	0.9967033	0.99670446	0.99670376	0.996704
d_1	0.13003055	0.13137588	0.13056468	0.13084639
x	−1.48485233	1.15791302	−0.4364434	0.11772183
y	−0.39946392	1.00188386	0.15693548	0.45082482
i_1	0.00466846	0.00466819	0.00466835	0.0046683
l_1	0.53552115	0.53580193	0.53571408	0.53577271
$x'(t \pm 0.25h)$	0.51119288	0.51145248	0.51158452	0.51156302
$y'(t \pm 0.25h)$	0.27151022	0.27085576	0.27138506	0.2712357

Now we are ready for a second approximation using the new elements at each time and the new values of p:

	P_1	P_4	P_2	P_3
T_0	15 : 42 : 00	20 : 52 : 00	17 : 45 : 00	18 : 50 : 00
m_0	1.53764686	1.53118707	0.46380123	0.46594147
M_0	−1.83360014	0.8575154	−1.22561432	0.25542194
n	0.57882291	0.57874561	0.57911016	0.57902118
N	1.08255544	1.08376454	1.08306354	1.08327401
(For reference) p	0.999777311	0.99858559	0.99962314	0.99690667
$\sin(\psi)$	−0.22387431	−0.22385595	−0.73972339	−0.74415818
ψ	3.36738053	−0.22576904	3.97425185	−0.83927375
τ	0.00416269	0.00567428	−0.00027673	−0.01235072
$T_0 + \tau = T$	15 : 42 : 15	20 : 52 : 20	17 : 44 : 59	18 : 49 : 15

Evidently, the first approximation is correct to a few seconds: the first approximation is good enough for most reasonable cases. We will not calculate new Besselian elements for the new times, we will just use the ones after the first approximation. From here:

(For reference) N	1.08255544	1.08376454	1.08306354	1.08327401
(For reference) ψ	3.36738053	-0.22576904	3.97425185	-0.83927375
$N + \psi = \gamma$	4.44993597	0.8579955	5.05731539	0.24400026
γ' (9.58)	-1.83407792	0.85636199	-1.22481794	0.24322743
$\sin(\gamma') = \xi$	-0.96554114	0.75546398	-0.94074412	0.24083631
$\cos(\gamma') = \eta_1$	-0.26025046	0.65519018	0.33911725	0.97056575
ϕ_1 (9.29)	-0.26100683	0.70698473	0.34291188	1.29522234
θ (9.29)	-1.53586099	1.68392198	-1.61769396	2.05487427
ϕ (9.30)	$-15° \, 0' \, 9''$	$40° \, 36' \, 8''$	$19° \, 42' \, 30''$	$74° \, 15' \, 40''$
λ (9.30)	$216° \, 36' \, 36''$	$323° \, 34' \, 7''$	$181° \, 9' \, 46''$	$15° \, 19' \, 52''$

Our answers can be arranged in a table:

Solar Eclipse of April 8, 2024 – Contacts of Penumbra

Event	Contact	Time	Latitude	Longitude
First External Contact	P_1	15 : 42 : 15	$-15° \, 0' \, 9''$	$216° \, 36' \, 36''$
First Internal Contact	P_2	17 : 44 : 59	$19° \, 42' \, 30''$	$181° \, 9' \, 46''$
Last Internal Contact	P_3	18 : 49 : 15	$74° \, 15' \, 40''$	$15° \, 19' \, 52''$
Last External Contact	P_4	20 : 52 : 20	$40° \, 36' \, 8''$	$323° \, 34' \, 7''$

We could make a third and more accurate approximation, but for the sake of brevity I will stop at two. For another method of calculation of the internal contacts, see section 9.14.

Only now that we know that this eclipse began globally at 15 : 42 and ended at 20 : 52, can we calculate the Besselian elements for the entire eclipse and tabulate them. The elements for the times in between the tabulated values can then be found via simple interpolation. If one is writing a program to predict eclipses however, tabulation and interpolation is unnecessary, simply start at the beginning time, calculate the elements and the coordinates of the location of the shadow of the Moon at that time (especially the special locations to be discussed hereafter), then increment the time by a preselected Δt (the smaller the better).

The Besselian elements for this eclipse, calculated at 15 minute intervals using methods detailed previously, can be found in section 9.18.

9.8 Rising and Setting Limits

In the last section we discussed how at the moment of contact at the location of contact, the shadow cone is tangent to the Earth and therefore the eclipse in those locations would be on the horizon. In particular, the contacts are:

1. First External Contact
   - The first place on the Earth at which the eclipse begins at sunrise.
2. First Internal Contact
   - The last place on the Earth at which the eclipse ends at sunrise.
3. Last Internal Contact
   - The first place on the Earth at which the eclipse begins at sunset.
4. Last External Contact
   - The last place on the Earth at which the eclipse ends at sunset.

Between the contacts, there are a whole set of points where the eclipse either begins or ends exactly at sunrise or sunset: returning to equation 9.52, we can write equation 9.18 as:

$$\begin{aligned}(l - i\zeta)\sin(Q) &= m\sin(M) - p\sin(\gamma)\\(l - i\zeta)\cos(Q) &= m\cos(M) - p\cos(\gamma)\end{aligned}$$

From the fact that $\zeta_1 = 0$ when the eclipse is on the horizon, we can write $\zeta = \zeta_1 = 0$ and therefore:

$$\left.\begin{aligned}l\sin(Q) &= m\sin(M) - p\sin(\gamma)\\l\cos(Q) &= m\cos(M) - p\cos(\gamma)\end{aligned}\right\} \quad (9.60)$$

If we add the squares of both equations, we get:

$$\begin{aligned}l^2 &= m^2 - 2mp\sin(M)\sin(\gamma) - 2mp\cos(M)\cos(\gamma) + p^2\\l^2 &= m^2 + p^2 - 2mp\cos(M-\gamma)\\l^2 - m^2 - p^2 &= -2mp\cos(M-\gamma)\\l^2 - m^2 - p^2 + 2mp &= -2mp\cos(M-\gamma) + 2mp\\\frac{l^2 - (m-p)^2}{2mp} &= 1 - \cos(M-\gamma)\\\frac{l^2 - (m-p)^2}{2mp} &= 2\sin^2((M-\gamma)/2)\\\frac{(l-m+p)(l+m-p)}{4mp} &= \sin^2((M-\gamma)/2)\end{aligned}$$

If we now set $\epsilon = M - \gamma$, we have:

$$\sin(\epsilon/2) = \pm\sqrt{\frac{(l-m+p)(l+m-p)}{4mp}} \quad (9.61)$$

Where $\epsilon/2$ can always be taken as less than 90°. Once we have ϵ, we can get γ by:

$$\gamma = M + \epsilon \quad (9.62)$$

And then carry on with equations 9.58 and 9.59 to get the latitude and longitude of the place. Now, because of the double sign on equation 9.61, we can see that there are two points where the eclipse is beginning or ending on the horizon at any moment in time between the internal and external contacts. Determining whether the Sun is rising or setting at the point calculated can be done by comparing θ, the hour angle of the eclipse. Remembering that the hour angle is measured such that West is positive, if the hour angle θ is less than 180°, the eclipse is happening at sunset. If the hour angle is more than 180° (or between −180° and 0°), the eclipse is happening at sunrise. Determining if the eclipse is beginning or ending can be done just like example 9.4, but since we have $\zeta = 0$, equation 9.50 simplifies to:

$$P' = a' + c' \sin(Q) - b' \cos(Q) \tag{9.63}$$

Where we have Q by equation 9.60.

Because the points at which the eclipse is in progress during sunrise or sunset can only exist between external and internal contact, if there are no internal contacts during an eclipse (as is frequently so), then there will always be points where the eclipse is in progress at the horizon and therefore the lines of beginning/ending at sunrise/sunset extend throughout the whole eclipse. If internal contacts exist, the points where the eclipse begins/ends at sunrise and the points at which the eclipse begins/ends at sunset form two distinct areas on the globe.

Example 9.6 Determine the rising and setting limits of the solar eclipse of April 8, 2024.

Solution

Because we have internal contacts in this eclipse, the rising / setting limits will form two distinct areas:

- The sunrise lobe between P_1 and P_2
- The sunset lobe between P_3 and P_4

I will give the calculation of the limiting points for the time 16 : 00 in full.

For this time, we have:

$$\begin{aligned}
\mu &= 1.04533016 \\
x &= m\sin(M) = -1.3314264 \\
y &= m\cos(M) = -0.31802844 \\
l_1 &= 0.53555609 \\
\rho_1 &= 0.99670336 \\
d_1 &= 0.13010879
\end{aligned}$$

From this we can deduce, in the same manner as example 9.5:

$$\begin{aligned}
m &= 1.36888215 \\
M &= -1.80526591
\end{aligned}$$

Using $p = 1$ as a first estimate, we can calculate ϵ via equation 9.61:

$$\begin{aligned}
\sin(\epsilon/2) &= \pm\sqrt{\frac{(0.53555609 - 1.36888215 + 1)(0.53555609 + 1.36888215 - 1)}{4 \cdot 1.36888215}} \\
&= \pm 0.16592441 \\
\therefore \epsilon_1 &= 2\arcsin(0.16592441) = 0.33339068 \\
\therefore \epsilon_2 &= 2\arcsin(-0.16592441) = -0.33339068
\end{aligned}$$

Now, we can find γ with equation 9.62, then γ' and then a better approximation for p with equation 9.58:

	$\epsilon > 0$	$\epsilon < 0$
$M + \epsilon = \gamma$	−1.47187523	−2.13865659
γ'	−1.47155018	−2.14015486
p	0.99996769	0.99904305

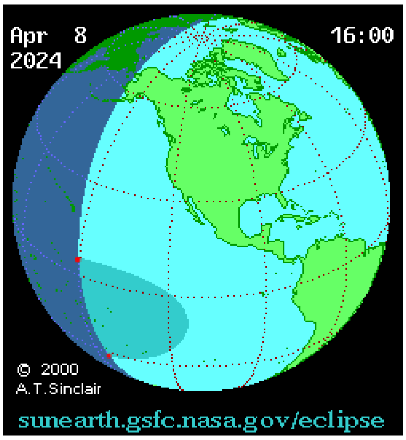
Now we substitute this value of p into equation 9.61:

$\sin(\epsilon/2)$	0.16591397	−2.13865659
ϵ	0.33336951	−0.33276149
$M + \epsilon = \gamma$	−1.4718964	−2.1380274
γ'	−1.47157142	−2.13952479

And now with this sufficiently accurate value of γ' we can get the longitude and latitude of the point with equations 9.59, 9.29, and 9.30:

$\sin(\gamma') = \xi$	−0.99508125	−0.84258646
$\cos(\gamma') = \eta_1$	0.09906216	−0.53856111
ϕ_1	0.0983835	−0.56333528
θ	−1.58371166	−1.48805765
	Sunrise	Sunrise
ϕ	$5° 39' 21''$	$-32° 21' 49''$
λ	$209° 22' 1''$	$214° 50' 51''$

These points we just calculated are these two points (marked red) on this map:



The fact that they are both for sunrise is to be expected since 16 : 00 is between P_1 and P_2 and therefore falls in the sunrise lobe.

To determine whether the eclipse is beginning or ending at these points, we have (for 16 : 00):

$$a'_1 = -0.00172705$$
$$b'_1 = -0.31653541$$
$$c'_1 = 0.50114349$$

Thus:

	$\epsilon > 0$	$\epsilon < 0$
$p\sin(\gamma)$	−0.99508123	−0.84258487
$p\cos(\gamma)$	0.09873559	−0.53678465
$x - p\sin(\gamma) = l\sin(Q)$ (9.60)	−0.33634517	−0.48884153
$y - p\cos(\gamma) = l\cos(Q)$ (9.60)	−0.41676403	0.21875621
Q	−2.46257385	−1.15002398
P' (9.63)	−0.56278455	−0.32986385
$P' < 0$ and $l > 0$	Beginning	Beginning

Note that here, we could have gotten a more accurate value for Q by calculating a more accurate value for p using the value of γ' we calculated earlier, but this is wholly unnecessary when only the algebraic sign of P' is necessary. (In fact, the fact that we did not do this correction is why $p\sin(\gamma)$ very slightly differs from the value of ξ we found earlier.)

In the same manner the following table is computed:

Solar Eclipse of April 8, 2024 – Rising and Setting Limits – Sunrise Lobe

Time	Latitude	Longitude			Latitude	Longitude	
15 : 42 : 15	−15° 0′	216° 37′	Begins at Sunrise				
15 : 45	−7° 6′	214° 47′	Begins at Sunrise		−22° 23′	216° 56′	Begins at Sunrise
16 : 00	5° 39′	209° 22′	Begins at Sunrise		−32° 22′	214° 51′	Begins at Sunrise
16 : 15	13° 26′	204° 34′	Begins at Sunrise		−36° 46′	211° 57′	Begins at Sunrise
16 : 30	19° 36′	199° 56′	Begins at Sunrise		−38° 41′	208° 36′	Begins at Sunrise
16 : 45	24° 45′	195° 24′	Begins at Sunrise		−38° 24′	204° 48′	Ends at Sunrise
17 : 00	29° 1′	190° 57′	Begins at Sunrise		−35° 29′	200° 27′	Ends at Sunrise
17 : 15	32° 15′	186° 37′	Begins at Sunrise		−28° 45′	195° 28′	Ends at Sunrise
17 : 30	33° 29′	182° 38′	Ends at Sunrise		−15° 34′	189° 41′	Ends at Sunrise
17 : 44 : 59	19° 43′	181° 10′	Ends at Sunrise				

Solar Eclipse of April 8, 2024 – Rising and Setting Limits – Sunset Lobe

Time	Latitude	Longitude			Latitude	Longitude	
18 : 49 : 15	74° 16′	15° 20′	Begins at Sunset				
19 : 00	46° 5′	352° 56′	Begins at Sunset		82° 30′	69° 48′	Begins at Sunset
19 : 15	30° 7′	345° 43′	Begins at Sunset		82° 28′	64° 46′	Ends at Sunset
19 : 30	21° 48′	340° 36′	Begins at Sunset		81° 39′	41° 11′	Ends at Sunset
19 : 45	17° 50′	336° 16′	Begins at Sunset		79° 15′	17° 39′	Ends at Sunset
20 : 00	16° 46′	332° 22′	Begins at Sunset		75° 20′	0° 14′	Ends at Sunset
20 : 15	18° 1′	328° 47′	Ends at Sunset		70° 6′	347° 39′	Ends at Sunset
20 : 30	21° 34′	325° 34′	Ends at Sunset		63° 19′	337° 46′	Ends at Sunset
20 : 45	28° 49′	323° 0′	Ends at Sunset		53° 23′	329° 3′	Ends at Sunset
20 : 52 : 20	40° 36′	323° 34′	Ends at Sunset				

9.9 Curve of Maximum Eclipse on the Horizon

While some places experience the beginning or end of the eclipse exactly at sunrise or sunset, some others experience maximum obscuration at sunrise or sunset. As discussed in section 9.6, the condition for maximum eclipse is:

$$P' = 0 \tag{9.64}$$

Expanding out P' for eclipse on the horizon ($\zeta \approx \zeta_1 = 0$), we have:

$$a' + c' \sin(Q) - b' \cos(Q) = 0 \tag{9.65}$$

This equation turns out to have two solutions:

$$\left.\begin{aligned} Q_1 &= 2 \arctan\left(\frac{\sqrt{-a'^2 + b'^2 + c'^2} - c'}{a' + b'}\right) \\ Q_2 &= 2 \arctan\left(\frac{-\sqrt{-a'^2 + b'^2 + c'^2} - c'}{a' + b'}\right) \end{aligned}\right\} \tag{9.66}$$

Now, let us say that the distance of the observer from the shadow axis at the moment of maximum eclipse is Δ, which obviously must be less than or equal to $|l|$ as the observer must be within the shadow. Then, we can say that at maximum eclipse:

$$\Delta \sin(Q) = x - \xi$$
$$\Delta \cos(Q) = y - \eta$$

In the above equation, we know x, y, and E, but we do not know Δ. Putting

$$\begin{cases} m \sin(M) = x \\ m \cos(M) = y \end{cases} \qquad \begin{cases} p \sin(\gamma) = \xi \\ p \cos(\gamma) = \eta \end{cases}$$

as before, the above conditions become:

$$\Delta \sin(Q) = m \sin(M) - p \sin(\gamma)$$
$$\Delta \cos(Q) = m \cos(M) - p \cos(\gamma)$$

And if we now subtract Q from all the angles, we get:

$$0 = m \sin(M - Q) - p \sin(\gamma - Q)$$
$$\Delta = m \cos(M - Q) - p \cos(\gamma - Q)$$

Now, if we put $\psi = \gamma - Q$, we have:

$$\left.\begin{aligned} \sin(\psi) &= \frac{m \sin(M - Q)}{p} \\ \Delta &= m \cos(M - Q) - p \cos(\psi) \end{aligned}\right\} \tag{9.67}$$

Where Δ must be positive as it is a distance. Then, we have:

$$\gamma = Q + \psi \tag{9.68}$$

From where we can carry on with equations 9.58 and 9.59 to get the longitude and latitude of the place.

Evidently, $\sin(\psi) = m \sin(M - Q)/p$ cannot be greater than 1 or less than -1. Here, the approximation $p = 1$ can be made and we can say that in order for points on this curve to exist, $m \sin(M - Q)$ must be less than 1 and greater than -1. Also, the first equation in 9.67 produces two values for ψ: the value of ψ that produces $\Delta > |l|$ must be discarded.

To find determine at which times the two conditions ($m \sin(M - Q)/p < 1$ and $\Delta \leq |l|$) are met, we must consider the extreme points of this curve. The extremes are the places where $\Delta = |l|$: i.e. the place experiences maximum eclipse when the place is on the boundary of the shadow of the Moon. This means that the Moon just barely grazes past the Sun in these locations and therefore are the edges of visibility of the eclipse.

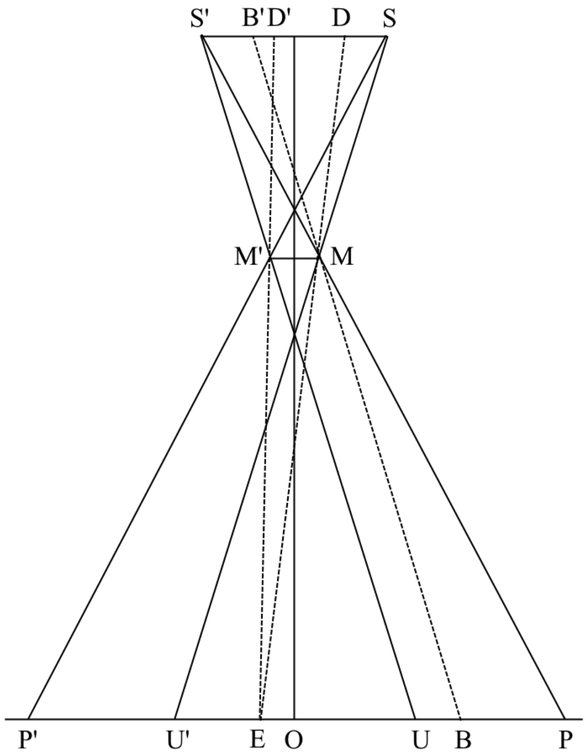
When $\Delta = l$ and we have the fact that the eclipse is on the horizon, our problem is simply the problem from the last section. Therefore the extreme points of the curve of maximum eclipse on the horizon must be on the sunrise and sunset curves. Since P' determines whether the eclipse is beginning, ending, or at maximum, we can look at our table of rising and setting limits (the table calculated in example 9.6), and determine when on the rising / setting limit curves the eclipse goes from beginning to ending (or vice versa).

If we say that at time t_1, $P' = P'_1$, at time t_2, $P' = P'_2$, and the sign of P'_1 is the opposite of the sign of P'_2, we can find the approximate time that P' flipped sign by approximating P' as linear in this portion. If we let t_0 be the time when $P' = 0$, we find:

$$t_0 = \frac{t_2 P'_1 - t_1 P'_2}{P'_1 - P'_2} \tag{9.69}$$

This approximation can be refined recursively, by finding the true value of P' at t_0 and repeating the calculations (similar to the bisection method used in chapter 4). Another way, that avoids recursive calculation, is to simply have a smaller time step which is not difficult computationally when done on a computer.

Once we have the time when $P' = 0$ on the rising/setting limits, we can simply calculate the place via the method of example 9.6. These will be the extreme points of the curve of maximum on the horizon.



To find the degree of obscuration, called the *magnitude* of the eclipse, at any point, consider this diagram, which is a simplified diagram of an annular eclipse of the Sun (the segment SS') by the Moon (the segment MM'). The point of observation B is outside the umbral shadow (region $U'U$), and within the penumbral shadow (region $P'P$), and sees a partial eclipse.

The known distances are: the radius of the penumbral shadow $OP = OP' = L_1$, the radius of the umbral shadow $OU = OU' = L_2$, and the distance from the center of the shadow to the observer $OB = \Delta$.

The magnitude is given by the amount of the Sun's diameter covered by the Moon. Therefore it is given by $S'B'/S'S$. By simple geometry, $S'B'$ corresponds to the segment PB on the ground (consider the triangles $S'B'M$ and MBP), and $S'S$ corresponds to the segment PU' (consider the triangles $S'SM$ and MPU). Thus, $S'B'/S'S = PB/PU'$, which is given by:

$$\text{Magnitude} = \frac{S'B'}{S'S} = \frac{L_1 - \Delta}{L_1 + L_2} \tag{9.70}$$

When the eclipse is central however, as seen from E, the magnitude is $D'D/S'S = (S'D - S'D')/S'S$. These lengths projected onto the ground give:

$$\begin{aligned} \text{Magnitude} &= \frac{D'D}{S'S} = \frac{S'D - S'D'}{S'S} \\ &= \frac{PE - UE}{PU'} = \frac{PU}{PU'} \\ &= \frac{L_1 - L_2}{L_1 + L_2} \end{aligned} \tag{9.71}$$

Equation 9.71 works even for total eclipses, in which case the magnitude is greater than 1 because L_2 is negative.

In the problem of the points of maximum eclipse on the horizon, since $\zeta = 0$, equation 9.15 gives $L = l$ and therefore L_1 and L_2 may be subtituted for l_1 and l_2.

Example 9.7 Determine the curve of maximum eclipse on the horizon of the solar eclipse of April 8, 2024.

Solution

We must first find the extreme points on the curve of maximum eclipse on the horizon. Looking at our computations in the previous example, we find four points where the eclipse goes from beginning to ending:

Time	ϵ	P'
16 : 30	< 0	-0.02759005
16 : 45	< 0	0.12821353
17 : 15	> 0	-0.13869916
17 : 30	> 0	0.00472722
19 : 00	< 0	-0.05823664
19 : 15	< 0	0.10380897
20 : 00	> 0	-0.0227394
20 : 15	> 0	0.12931085

Now, using decimal hours (e.g. 16 : 30 = 16.5) as our values of t, we can find the approximate times P' flips sign for each of these intervals using equation 9.69:

$$t_{01} = \frac{16.75 \cdot (-0.02759005) - 16.5 \cdot 0.12821353}{-0.02759005 - 0.12821353} = 16:32:38$$
$$t_{02} = \frac{17.5 \cdot (-0.13869916) - 17.25 \cdot 0.00472722}{-0.13869916 - 0.00472722} = 17:29:30$$
$$t_{03} = \frac{19.25 \cdot (-0.05823664) - 19.0 \cdot 0.10380897}{-0.05823664 - 0.10380897} = 19:05:23$$
$$t_{04} = \frac{20.25 \cdot (-0.0227394) - 20.0 \cdot 0.12931085}{-0.0227394 - 0.12931085} = 20:02:15$$

We will not do a second approximation. (When predicting eclipses with a program, simply use smaller time steps than 15 minutes to get more accurate answers.)
Now we have to find the points on the rising and setting curve at these times. Using linearly interpolated values from the elements table in section 9.18:

t	d	μ	x	y	l_1	ρ_1	d_1
16 : 32 : 38	0.12981883	1.18785924	-1.05311448	-0.17027728	0.53561349	0.99670349	0.13025064
17 : 29 : 30	0.13006473	1.43597894	-0.56856575	0.08686272	0.53569566	0.9967037	0.13049733
19 : 05 : 23	0.13047918	1.8544714	0.24893948	0.52036319	0.53578202	0.99670406	0.13091313
20 : 02 : 15	0.13072501	2.10260536	0.73370105	0.77722511	0.53580264	0.99670427	0.13115976

(Note that calculating the true elements at each of these times from the ephemeris using the method of example 9.2 is even better and should be done when eclipse prediction is done using a program.)

We find our extreme points (using the method of example 9.6, but the sign of ϵ to be used is already known):

Point	Time	Latitude	Longitude
Sunrise Southern Extreme	16 : 32 : 38	$-38° 48' 13''$	$207° 58' 0''$
Sunrise Northern Extreme	17 : 29 : 30	$33° 29' 55''$	$182° 45' 29''$
Sunset Northern Extreme	19 : 05 : 23	$82° 31' 16''$	$72° 8' 45''$
Sunset Southern Extreme	20 : 02 : 15	$16° 49' 18''$	$331° 48' 27''$

Whether the point is northern or southern is determined by the value of $l\cos(Q)$ (see equation 9.10). The Q for these points was found in the same way as example 9.6: equation 9.60. If $l\cos(Q)$ is negative, the point is northern. If $l\cos(Q)$ is positive, the point is southern. If no internal contacts exist, there will only be two extremes, one per lobe, (as the sunrise and sunset lobes are connected, the curve of maximum eclipse on the horizon will extend throughout the whole eclipse), and both will be northern or both will be southern.

Now we know the curve of maximum eclipse on the horizon extends only from 16 : 32 : 38 to 17 : 29 : 30 for the sunrise lobe, and from 19 : 05 : 23 to 20 : 02 : 15 in the sunset lobe. The points in between are calculated via the method detailed above in this section: I will give the calculation for the point at time 16 : 45 in full.

At this time, we have:

$$
\begin{aligned}
m &= 0.95478937 \\
M &= -1.69092252
\end{aligned}
$$

and

$$
\begin{aligned}
a_1' &= -0.0012427 \\
b_1' &= -0.30358228 \\
c_1' &= 0.50818381
\end{aligned}
$$

Thus, we can calculate Q by equation 9.66:

$$
\begin{aligned}
Q_1 &= 2\arctan\left(\frac{\sqrt{-(-0.0012427)^2 + (-0.30358228)^2 + (0.50818381)^2} - 0.50114349}{-0.0012427 + (-0.30358228)}\right) \\
&= -0.53639649 \\
Q_2 &= 2\arctan\left(\frac{-\sqrt{-(-0.0012427)^2 + (-0.30358228)^2 + (0.50818381)^2} - 0.50114349}{-0.0012427 + (-0.30358228)}\right) \\
&= 2.60099756
\end{aligned}
$$

We find ψ and Δ with equation 9.67 for each value of Q, first approximating $p = 1$:

$$
\begin{aligned}
\sin(\psi)_1 &= (0.95478937\sin(-1.69092252 - (-0.53639649)))/1 &&= -0.87325361 \\
\psi_{11} &= \arcsin(-0.87325361) &&= -1.06184017 \\
\Delta_{11} &= 0.95478937\cos(-1.69092252 - (-0.53639649)) - 1 \cdot \cos(-1.06184017) &&= -0.10119491 \\
\psi_{12} &= \pi - \arcsin(-0.87325361) &&= 4.20343283 \\
\Delta_{12} &= 0.95478937\cos(-1.69092252 - (-0.53639649)) - 1 \cdot \cos(4.20343283) &&= 0.87333703
\end{aligned}
$$

$$
\begin{aligned}
\sin(\psi)_2 &= (0.95478937\sin(-1.69092252 - 2.60099756))/1 &&= 0.87162496 \\
\psi_{21} &= \arcsin(0.87162496) &&= 1.05850769 \\
\Delta_{21} &= 0.95478937\cos(-1.69092252 - 2.60099756) - 1 \cdot \cos(1.05850769) &&= -0.87990746 \\
\psi_{22} &= \pi - \arcsin(0.87162496) &&= 2.08308496 \\
\Delta_{22} &= 0.95478937\cos(-1.69092252 - 2.60099756) - 1 \cdot \cos(2.08308496) &&= 0.10043927
\end{aligned}
$$

Considering $l_1 = 0.53563338$ at this time, ψ_{22} (ψ_2 for Q_2) is the only ψ value that produces a positive Δ value that is less than l. Thus, we continue with ψ_{22} and Q_2.

$Q_2 + \psi_{22} = \gamma$	4.68408253
ρ_1	0.99670449
γ'	-1.59919632
p	0.99999735
$m\sin(M - Q_2)/p = \sin(\psi)_2$	0.87162727
$\pi - \arcsin(\sin(\psi))_2 = \psi_{22}$	2.08308025
$Q_2 + \psi_{22} = \gamma$	4.68407781
γ'	-1.59920106
$\sin(\gamma') = \xi$	-0.99959661
$\cos(\gamma') = \eta_1$	-0.02753202
ϕ_1	-0.02816386
θ	-1.56710457
	Sunrise
ϕ	$-1^\circ\ 37'\ 9''$
λ	$199^\circ\ 3'\ 56''$

The magnitude at this point is given by equation 9.70 since $\Delta = 0.10043927 > |l_2| = 0.01048206$ and therefore the eclipse only partial:

$$l_1 = 0.53563338$$
$$l_2 = -0.01048206$$
$$\text{Magnitude} = \frac{0.53563338 - 0.10043927}{0.53563338 + (-0.01048206)} = 0.82870231$$

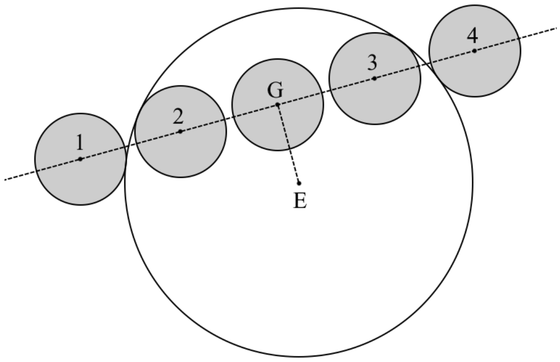
Since the extreme points are defined as places where $\Delta = l_1$, the magnitude at these points is always going to be 0%.

In this manner the following table is computed:

Solar Eclipse of April 8, 2024 – Curves of Maximum Eclipse on the Horizon

Time	Latitude	Longitude	Magnitude
16 : 32 : 38	$-38^\circ\ 48'$	$207^\circ\ 58'$	0%
16 : 45	$-1^\circ\ 37'$	$199^\circ\ 4'$	83%
17 : 00	$12^\circ\ 34'$	$193^\circ\ 26'$	43%
17 : 15	$23^\circ\ 50'$	$188^\circ\ 2'$	17%
17 : 29 : 30	$33^\circ\ 30'$	$182^\circ\ 45'$	0%
19 : 05 : 23	$82^\circ\ 31'$	$72^\circ\ 9$	0%
19 : 15	$79^\circ\ 44'$	$27^\circ\ 46'$	11%
19 : 30	$70^\circ\ 41'$	$359^\circ\ 36'$	34%
19 : 45	$58^\circ\ 21'$	$346^\circ\ 9'$	68%
20 : 00	$38^\circ\ 20'$	$336^\circ\ 3'$	73%
20 : 02 : 15	$16^\circ\ 49'$	$331^\circ\ 48'$	0%

9.10 Greatest Eclipse



In this diagram of the path of the Moon's shadow across the Earth, the point labeled G is called the point of *greatest eclipse*. It is the point at which the distance from the center to the Earth to the center of the shadow, a variable we have been calling m, is minimum.

Minimizing m means minimizing m^2, and since $m^2 = x^2 + y^2$, the derivative is:

$$\frac{d(m^2)}{dt} = 2xx' + 2yy'$$

Which needs to be 0.
Since we can linearly approximate x and y as:

$$x = x_0 + x'\tau$$
$$y = y_0 + y'\tau$$

The derivative of m^2 expands to:

$$\frac{d(m^2)}{dt} = 2(x_0x' + x'^2\tau) + 2(y_0y' + y'^2\tau)$$

Setting this to 0 and solving for τ gives:

$$\tau = -\frac{x_0x' + y_0y'}{x'^2 + y'^2} \tag{9.72}$$

And then T is given by $T = T_0 + \tau$.

Successive uses of equation 9.72 refine our estimation for the time of greatest eclipse. The value of m at this point, i.e. the minimum value of m, is called the *gamma* of the eclipse.

The location of greatest eclipse on the surface of the Earth can have two distinct cases. If central contacts exist (to be discussed in the next section), the point of greatest eclipse lies on the curve of centrality (to be discussed in the next section) at the time of greatest eclipse. If central contacts do not exist, the point of greatest eclipse lies on the curve of maximum eclipse on the horizon, at the time of greatest eclipse.

Example 9.8 Determine the time of greatest eclipse and the gamma of the solar eclipse of April 8, 2024.

Solution
We choose $T_0 = 18:00$ as a first approximation. At this time,

$$x_0 = -0.30856088$$
$$y_0 = 0.22479055$$
$$x' = 0.51147366$$
$$y' = 0.27129112$$

Thus, equation 9.72 gives:

$$\tau = -\frac{(-0.30856088) \cdot 0.51147366 + 0.22479055 \cdot 0.27129112}{0.51147366^2 + 0.27129112^2}$$
$$= 0.28888984h$$
$$T = 18:00 + 0.28888984h = 18:17:20$$

We now linearly interpolate for the elements for this time (again, calculating their true values would be even better):

t	x	y	x'	y'
18 : 17 : 20	−0.16080953	0.30312923	0.51154119	0.27119988

Now, equation 9.72 gives:

$$\tau = -\frac{(-0.16080953)\cdot 0.51154119 + 0.30312923\cdot 0.27119988}{0.51154119^2 + 0.27119988^2}$$
$$= 0.00015538h$$
$$T = 18:17:20 + 0.00015538h = 18:17:21$$

We will not make a third approximation.
The gamma is given by the $m = \sqrt{x^2+y^2}$ at this time. Since the second approximation makes such a small difference, we will just use the x and y for 18 : 70 : 20, listed in the table above:

$$\text{Gamma } = \sqrt{(-0.16080953)^2 + 0.30312923^2} = 0.34314288$$

9.11 Curve of Centrality

When the shadow of the Moon passes over the Earth, the line the center of the shadow makes on the Earth is called the *curve of centrality* or *central curve*. Therefore the points on this curve have distance 0 from the center of the shadow. Thus, equation 9.10 become the extremely simple:

$$\xi = x$$
$$\eta = y$$

Therefore, we have (by equation 9.23 and 9.24):

$$\left.\begin{aligned} \xi &= x \\ \eta_1 &= \frac{y}{\rho_1} \\ \zeta_1 &= 1 - \xi^2 - \eta_1^2 \end{aligned}\right\} \quad (9.73)$$

From which we continue with equations 9.29 and 9.30.

To find the times and locations at which the curve of centrality starts and ends, we can repeat the method of finding contacts in example 9.5 while setting $l = 0$. This makes the double sign in equation 9.55 obsolete and therefore there are only two contacts of the center with the Earth (if they even exist).

To find the duration of totality or annularity along the central curve, we use the fact that time is distance divided by speed and that time must be positive. Thus:

$$\text{Duration} = \left|\frac{2L_2}{v}\right| = \left|\frac{2(l_2 - i_2\zeta)}{v}\right| \quad (9.74)$$

Where v is simply the speed at which the place of observation is moving with respect to the shadow, or:

$$v^2 = (x' - \xi')^2 + (y' - \eta')^2 \quad (9.75)$$

With ξ' and η' being calculated by equation 9.40:

$$\xi' = \mu'(-\eta\sin(d) + \zeta\cos(d))$$
$$\eta' = \mu'\xi\sin(d) - d'\zeta$$

But since we have $\xi = x$ and $\eta = y$ on the central curve, we can write:

$$\left.\begin{aligned} \xi' &= \mu'(-y\sin(d) + \zeta\cos(d)) \\ \eta' &= \mu' x\sin(d) - d'\zeta \end{aligned}\right\} \quad (9.76)$$

Example 9.9 Determine the curve of centrality of the solar eclipse of April 8, 2024.

Solution

First we have to determine the central contacts. We repeat the method of example 9.5 but set $l = 0$ in equation 9.55. Taking $T_0 = 18:00$ and $p = 1$ as a first approximation, we find:

	First Contact	Last Contact
τ	-1.33343322	1.91121291
T	16 : 41 : 00	19 : 54 : 40
p	0.99993787	0.99817491

We now linearly interpolate for the elements for these times (again, calculating their true values would be even better):

t	d	μ	x	y	i_2	l_2	ρ_1
16 : 41 : 00	0.12985038	1.21988483	-0.99057474	-0.13707257	0.00464516	-0.01049008	0.99670351
19 : 54 : 40	0.13069238	2.06956364	0.66914899	0.74302244	0.00464499	-0.01031513	0.99670424
t	d_1	ρ_2	d_2	d'	μ'	x'	y'
16 : 41 : 00	0.13028229	0.99994388	0.12941987	0.00025897	0.26187053	0.5114177	0.27144902
19 : 54 : 40	0.13112703	0.99994315	0.13025915	0.00025898	0.26187148	0.51157069	0.27105629

Now repeating the calculations for the contacts, we obtain:

	First Contact	Last Contact
T_0	16 : 40 : 00	19 : 54 : 40
τ	0.00052253	-0.00321216
T	16 : 40 : 02	19 : 54 : 28
γ'	-1.70807333	0.73085298
ϕ	$-7° 49' 27''$	$47° 40' 46''$
λ	$201° 8' 2''$	$339° 41' 25''$

Now that we know the curve of centrality extends from 16 : 40 : 02 to 19 : 54 : 28, we can compute the curve. I will give the calculation for the point at time 18 : 00 in full.

At this time, the elements are:

t	d	μ	x	y	i_2	l_2	ρ_1
18 : 00	0.13019636	1.56907269	-0.30856088	0.22479055	0.0046451	-0.01038565	0.99670381
t	d_1	ρ_2	d_2	d'	μ'	x'	y'
18 : 00	0.13062939	0.99994358	0.12976473	0.00025898	0.26187054	0.51147366	0.27129112

equation 9.73 gives:

$$\begin{aligned} \xi &= -0.30856088 \\ \eta_1 &= \frac{0.22479055}{0.99670381} &&= 0.22553395 \\ \zeta_1 &= \sqrt{1 - (-0.30856088)^2 - (0.22553395)^2} &&= 0.92408042 \end{aligned}$$

Then, equations 9.29 and 9.30 give:

$$\begin{aligned}\phi_1 &= 0.35115385\\ \theta &= -0.33483579\\ \phi &= 20^\circ\ 10'\ 51''\\ \lambda &= 250^\circ\ 54'\ 51''\end{aligned}$$

Now to find the central duration. Equation 9.27 gives ζ:

$$\begin{aligned}\zeta &= -0.99994358 \cdot 0.22553395 \sin(0.13062939 - 0.12976473)\\ &\quad + 0.99994358 \cdot 0.92408042 \cos(0.13062939 - 0.12976473)\\ &= 0.92383294\end{aligned}$$

Equation 9.76 gives ξ' and η':

$$\begin{aligned}\xi' &= 0.26187054(-0.22479055 \sin(0.13019636) + 0.92383294 \cos(0.13019636))\\ &= 0.23223457\\ \eta' &= 0.26187054 \cdot (-0.30856088) \sin(0.13019636) - 0.00025898 \cdot 0.92383294\\ &= -0.0107298\end{aligned}$$

Now, equation 9.75 gives v:

$$\begin{aligned}v^2 &= (0.51147366 - 0.23223457)^2 + (0.27129112 - (-0.0107298))^2 &= 0.15751026\\ \therefore v &= \sqrt{0.15751026} &= 0.39687563\end{aligned}$$

Then equation 9.74 gives the central duration:

$$\text{Duration} = \left| \frac{2(-0.01038565 - 0.0046451 \cdot 0.92383294)}{0.39687563} \right| = 0.07396245h$$

The result is in hours because all our derivatives are in units of hours. When converted to minutes and seconds, the duration is:

$$\text{Duration} = 4m\ 26.3s$$

When calculating the duration of central eclipse for the central contact points, one might run into the issue of the interpolated values giving impossible values for ζ_1. Simply use $\zeta_1 = 0$ for those points, as this is the theoretically correct value. (These points correspond to the points that experience maximum centrality on the horizon, so $\zeta_1 = 0$.)

In this manner the following table is computed:

Solar Eclipse of April 8, 2024 – Curve of Centrality

Time	Latitude	Longitude	Central Duration
16 : 40 : 02	$-7°\ 49'$	$201°\ 8'$	$2m\ 7s$
16 : 45	$-4°\ 20'$	$216°\ 56'$	$2m\ 42s$
17 : 00	$1°\ 36'$	$230°\ 0'$	$3m\ 20s$
17 : 15	$6°\ 38'$	$237°\ 11'$	$3m\ 46s$
17 : 30	$11°\ 18'$	$242°\ 28'$	$4m\ 5s$
17 : 45	$15°\ 48'$	$246°\ 53'$	$4m\ 18s$
18 : 00	$20°\ 11'$	$250°\ 55'$	$4m\ 26s$
18 : 15	$24°\ 29'$	$254°\ 54'$	$4m\ 30s$
18 : 17 : 21	$25°\ 9'$	$255°\ 32'$	$4m\ 30.2s$
18 : 30	$28°\ 46'$	$259°\ 9'$	$4m\ 29s$
18 : 45	$33°\ 0'$	$263°\ 59'$	$4m\ 23s$
19 : 00	$37°\ 13'$	$269°\ 50'$	$4m\ 12s$
19 : 15	$41°\ 21'$	$277°\ 27'$	$3m\ 55s$
19 : 30	$45°\ 17'$	$288°\ 13'$	$3m\ 31s$
19 : 45	$48°\ 33'$	$306°\ 3'$	$2m\ 56s$
19 : 54 : 28	$47°\ 41'$	$339°\ 41'$	$2m\ 5s$

In theory, the point of greatest eclipse will experience the longest central eclipse in the case of total eclipse. In practice however, because of the various mountains and valleys on the Moon, the point of greatest duration might differ from the point of greatest eclipse by a few degrees in coordinates. This is entirely too difficult to calculate, and because the duration difference will only be of a tiny fraction of a second, we will not go into the calculation of that here.

For annular eclipses, the point of greatest duration may be on the central line near the sunrise or sunset lobes, instead of at greatest eclipse.

9.12 Limits of Partiality

There are generally two limits at which partial eclipses can be observed, one in the North and one in the South. In the case that there are no internal contacts, only one of the two will exist. As discussed in section 9.9, The limit of visibility of an eclipse is given by two conditions:

$$P' = 0$$
$$\Delta = l$$

Which means that the moment of maximum eclipse occurs when the point of observation is just on the edge of the shadow – meaning the shadow of the Moon only barely grazes past this point. To solve the first of these conditions, recall the expression for P' (equation 9.50):

$$P' = a' + (c' - (i^2+1)\zeta\mu' \cos(d))\sin(Q) + (-b' + (i^2+1)\zeta d')\cos(Q)$$

In this equation, if we set:

$$\left.\begin{aligned} C &= c' - (i^2+1)\zeta\mu' \cos(d) \\ B &= b' - (i^2+1)\zeta d' \end{aligned}\right\} \quad (9.77)$$

Then, we have for P':

$$P' = a' + C\sin(Q) - B\cos(Q)$$

Which is of the same form as equation 9.65, and thus the solutions are:

$$\left.\begin{aligned} Q_1 &= 2\arctan\left(\frac{\sqrt{-a'^2 + B^2 + C^2} - C}{a' + B}\right) \\ Q_2 &= 2\arctan\left(\frac{-\sqrt{-a'^2 + B^2 + C^2} - C}{a' + B}\right) \end{aligned}\right\} \quad (9.78)$$

We need to find ζ to find C and B. In order to do this, we first assume $\zeta = 0$ and find two values of Q by equation 9.77 and 9.78. Then, equation 9.28∗ gives:

$$\left.\begin{aligned} \xi &= x - (l - i\zeta)\sin(Q) \\ \eta_1 &= (y - (l - i\zeta)\cos(Q))/\rho_1 \\ \zeta_1^2 &= 1 - \xi^2 - \eta_1^1 \end{aligned}\right\} \quad (9.28^*)$$

From which equation 9.27 gives a second approximation of ζ:

$$\zeta = -\rho_2\eta_1 \sin(d_1 - d_2) + \rho_2\zeta_1 \cos(d_1 - d_2) \quad (9.27)$$

Repeatedly applying equations 9.28∗ and 9.27 until convergence, we can now use the final value in equation 9.77 and 9.78 for the accurate value of Q. Once Q has been found, apply equation 9.28∗ once more, from which equation 9.29 and 9.30 give the latitude and longitude of the place.

Since there are two values of Q, two points can be found. If $(l - i\zeta)\cos(Q)$ for that point is negative, the point belongs to the northern limit. If $(l - i\zeta)\cos(Q)$ is positive, it belongs to the southern limit. The extreme points of these curves are the points where $\zeta = 0$, which we have already found as the northern and southern extremes of the curve of maximum eclipse on the horizon. The northern limit of partiality will connect the two northern extremes of the curve of maximum eclipse on the horizon, and the southern limit will connect the two southern extremes.

Example 9.10 Determine the limits of partiality of the solar eclipse of April 8, 2024.

Solution
Recall from example 9.7 when we computed the extremes of the curve of maximum eclipse on the horizon:

Point	Time	Latitude	Longitude
Sunrise Southern Extreme	16 : 32 : 38	−38° 48′ 13″	207° 58′ 0″
Sunrise Northern Extreme	17 : 29 : 30	33° 29′ 55″	182° 45′ 29″
Sunset Northern Extreme	19 : 05 : 23	82° 31′ 16″	72° 8′ 45″
Sunset Southern Extreme	20 : 02 : 15	16° 49′ 18″	331° 48′ 27″

These are also the extremes of the limits of partiality. Thus the northern limit of partiality extends from 17 : 29 : 30 to 19 : 05 : 23, and the southern limit extends from 16 : 32 : 38 to 20 : 02 : 15. I will give the solution for the time 18 : 00 in full.

For this time, we have:

$$a_1' = -0.00043531 \qquad b_1' = -0.28178103 \qquad c_1' = 0.51976556$$

and:

$$x = -0.30856088 \qquad y = 0.22479055 \qquad i_1 = 0.00466835 \qquad l_1 = 0.53573027$$

We first assume $\zeta = 0$. We calculate B and C by equation 9.77:

$$\begin{aligned} B &= -0.28178103 - (0.00466835^2 + 1)\cdot 0 \cdot 0.00025898 &&= -0.28178103 \\ C &= 0.51976556 - (0.00466835^2 + 1)\cdot 0 \cdot 0.26187054\cos(0.13019636) &&= 0.51976556 \end{aligned}$$

Now, equation 9.78 gives the first approximate values of Q:

$$Q_1 = 2\arctan\left(\frac{\sqrt{-(-0.00043531)^2 + (-0.28178103)^2 + 0.51976556^2} - 0.51976556}{-0.00043531 + (-0.28178103)}\right)$$
$$= -0.49604542$$
$$Q_2 = 2\arctan\left(\frac{-\sqrt{-(-0.00043531)^2 + (-0.28178103)^2 + 0.51976556^2} - 0.51976556}{-0.00043531 + (-0.28178103)}\right)$$
$$= 2.64407467$$

Using Q_1 in equation 9.28∗ gives:

$$\begin{aligned}
\xi &= -0.30856088 - (0.53573027 - 0.00466835 \cdot 0)\sin(-0.49604542) \\
&= -0.05357934 \\
\eta_1 &= (0.22479055 - (0.53573027 - 0.00466835 \cdot 0)\cos(-0.49604542))/0.99670381 \\
&= -0.24718378 \\
\zeta_1 &= \sqrt{1 - (-0.05357934)^2 - (-0.24718378)^2} \\
&= 0.96748614
\end{aligned}$$

Which then put into 9.27 gives:

$$\begin{aligned}
\zeta = &-0.99994358 \cdot -0.24718378\sin(0.13062939 - 0.12976473) \\
&+ 0.99994358 \cdot 0.96748614\cos(0.13062939 - 0.12976473) \\
= &\ 0.96764491
\end{aligned}$$

This value substituted into equation 9.77 gives:

$$\begin{aligned}
B &= -0.28178103 - (0.00466835^2 + 1) \cdot 0.96764491 \cdot 0.00025898 && = -0.28203162 \\
C &= 0.51976556 - (0.00466835^2 + 1) \cdot 0.96764491 \cdot 0.26187054\cos(0.13019636) && = 0.26850704
\end{aligned}$$

Now, equation 9.78 gives the second approximate values of Q:

$$Q_1 = 2\arctan\left(\frac{\sqrt{-(-0.00043531)^2 + (-0.28203162)^2 + 0.26850704^2} - 0.26850704}{-0.00043531 + (-0.28203162)}\right)$$
$$= -0.80884143$$

Which in 9.28∗ gives:

$$\begin{aligned}
\xi &= 0.07576525 \\
\eta_1 &= -0.14239483 \\
\zeta_1 &= 0.98690594
\end{aligned}$$

Which in 9.27 gives a second approximate value of ζ:

$$\zeta = 0.98697301$$

The table of repetitions is given here:

Repetition	ζ (Q_1)	ζ (Q_2)
1	0.96764491	0.44051669
2	0.98697301	0.42532847
3	0.98723346	0.42587744
4	0.98723688	0.4258576
5	0.98723692	0.42585832
6	0.98723692	0.42585829
7	0.98723692	0.42585829

Now that ζ has converged, we can continue:

	ζ (Q_1)	ζ (Q_2)
ζ	0.98723692	0.42585829
B	-0.28203669	-0.28189132
C	0.26341978	0.40918726
Q	-0.81838782	2.53747552
$(l - i\zeta)\cos(Q)$	$0.36296793 > 0$	$-0.43927193 < 0$
	Southern	Northern
ξ	0.07918173	-0.61174551
η_1	-0.13863435	0.6662586
ζ_1	0.98717312	0.42645857
ϕ_1	-0.00886593	0.79824443
θ	0.07926783	-1.06848099
ϕ	$-0° 30' 35''$	$45° 49' 56''$
λ	$274° 38' 26''$	$208° 52' 46''$

In this manner the following table is computed:

Solar Eclipse of April 8, 2024 – Limits of Partial Eclipse

Time	Northern Latitude	Northern Longitude		Southern Latitude	Southern Longitude
16 : 32 : 38				$-38° 48'$	$207° 58'$
16 : 45				$-25° 56'$	$247° 3'$
17 : 00				$-18° 59'$	$256° 43'$
17 : 15				$-13° 23'$	$262° 49'$
17 : 29 : 30	$33° 30'$	$182° 45'$		(1)	(1)
17 : 30	(2)	(2)		$-8° 36'$	$267° 23'$
17 : 45	$39° 58'$	$201° 43'$		$-4° 19'$	$271° 12'$
18 : 00	$45° 50$	$208° 53'$		$-0° 31'$	$274° 38'$
18 : 15	$52° 9'$	$214° 10'$		$2° 55'$	$277° 57'$
18 : 30	$59° 14'$	$218° 53'$		$6° 2'$	$281° 19'$
18 : 45	$68° 7'$	$223° 53'$		$8° 52'$	$284° 56'$
19 : 00	$81° 32'$	$230° 2'$		$11° 28'$	$288° 59'$
19 : 05 : 23	$82° 31'$	$72° 9'$		(1)	(1)
19 : 15				$13° 48'$	$293° 46'$
19 : 30				$15° 51'$	$299° 50'$
19 : 45				$17° 26'$	$308° 19'$
20 : 00				(2)	(2)
20 : 02 : 15				$16° 49'$	$331° 48'$

[1] These points were not computed for the sake of brevity.
[2] While these points should exist, our numbers produce impossible results. This is due to the fact that these points are very close to the edges of the curve and our derivatives are not good enough (recall that we took 15 minute time steps for the derivatives). With a smaller time step, and thus more accurate derivatives, this problem should disappear.

9.13 Limits of Centrality

The limits of centrality are obtained exactly like the limits of partiality: simply replace the penumbral variables i_1, l_1, a'_1, b'_1, c'_1 with the umbral variables i_2, l_2, a'_2, b'_2, c'_2. The extremes of the limits can be obtained exactly as the extremes of the penumbral limits:

1. Compute the umbral contact U_1, U_2, U_3, U_4. (Redo example 9.5 with umbral variables.)
2. Compute the rising / setting limits for the umbra, i.e. the curves of central eclipse beginning/ending on the horizon. (Redo example 9.6 with umbral variables.)
3. Compute the times and locations on the rising / setting limtis where P' flips sign. (Redo the first part of example 9.7 with umbral variables.)

While this more rigorous method is recommended, and even necessary for worldbuilding cases where the umbral shadow is significantly large, for cases similar to our Earth, where the umbra is very small, we would find that the umbral external and internal contacts are only a few minutes apart (U_1 and U_2, or U_3 and U_4, are both only 2 minutes apart for the case of the April, 8, 2024 eclipse), and therefore our time step of 15 minutes will not be able to capture any detail. While a smaller time step would resolve this issue, we have another workaround:

The extreme points lie on the rising / setting limits, and the extreme points lie on those limits at some time in between the external and internal contacts. Also, the time between the external and internal umbral contacts is so small, that the time between the extreme points at each lobe, which is even smaller, is absolutely tiny, to the point where they occur almost at the same instant (less than a minute apart for the case of the April, 8, 2024 eclipse). In addition, the area covered by these rising / setting limits is also very small. We can therefore say that the points on the rising / setting limits at the time midway between the contacts are close enough to the true extreme points. The time midway between the umbral contacts is well approximated by the time of contact of the center of the shadow, which we have already calculated in example 9.9. Thus, the points on the rising / setting limits at the time of central contact can be used as the extreme points of the limits of centrality.

Very rarely (or more commonly when the umbra is very large), a total or annular eclipse will occur without the center of the shadow landing on the Earth (as an example, see the solar eclipse of April, 2043). In this case, the more rigorous method is needed.

Example 9.11 Determine the limits of totality of the solar eclipse of April 8, 2024.

Solution
We use the approximation for the extreme points. The times of central contact are 16 : 40 : 02 and 19 : 54 : 28. Using the elements at these times (interpolated in example 9.9, not exactly at the contact times but should still be sufficiently accurate):

t	d	μ	x	y	i_2	l_2	ρ_1
16 : 41 : 00	0.12985038	1.21988483	−0.99057474	−0.13707257	0.00464516	−0.01049008	0.99670351
19 : 54 : 40	0.13069238	2.06956364	0.66914899	0.74302244	0.00464499	−0.01031513	0.99670424
t	d_1	ρ_2	d_2	d'	μ'	x'	y'
16 : 41 : 00	0.13028229	0.99994388	0.12941987	0.00025897	0.26187053	0.5114177	0.27144902
19 : 54 : 40	0.13112703	0.99994315	0.13025915	0.00025898	0.26187148	0.51157069	0.27105629

we find, via the method of example 9.5 but swapping out l_1 for l_2:

Point	Time	Latitude	Longitude
Sunrise Northern Extreme	16 : 40 : 00	$-7°\ 15'\ 47''$	$201°\ 3'\ 33''$
Sunrise Southern Extreme	16 : 40 : 00	$-8°\ 27'\ 45''$	$201°\ 13'\ 8''$
Sunset Northern Extreme	19 : 54 : 40	$48°\ 12'\ 38''$	$339°\ 52'\ 46''$
Sunset Southern Extreme	19 : 54 : 40	$47°\ 4'\ 2''$	$339°\ 32'\ 42''$

Which are sufficiently accurate. For a more accurate computation, the rigorous method, the one we followed for the penumbra, must be used.

Now, we find the limits just like in example 9.10, simply swapping out the penumbral variables for umbral ones, and obtain the table below.

Solar Eclipse of April 8, 2024 – Limits of Total Eclipse

Time	Northern Latitude	Northern Longitude		Southern Latitude	Southern Longitude
16 : 41 : 00	$-7° 16'$	$201° 4'$		$-8° 28'$	$201° 13'$
16 : 45	$-3° 55'$	$215° 45'$		$-4° 45'$	$218° 5'$
17 : 00	$2° 6'$	$229° 7'$		$1° 6'$	$230° 50'$
17 : 15	$7° 9'$	$236° 23'$		$6° 6'$	$237° 59'$
17 : 30	$11° 51'$	$241° 41'$		$10° 46'$	$243° 15'$
17 : 45	$16° 22'$	$246° 7'$		$15° 15'$	$247° 39'$
18 : 00	$20° 46'$	$250° 10'$		$19° 36'$	$251° 40'$
18 : 15	$25° 6'$	$254° 10'$		$23° 43'$	$255° 37'$
18 : 30	$29° 24'$	$258° 27'$		$28° 7'$	$259° 50'$
18 : 45	$33° 41'$	$263° 21'$		$32° 19'$	$264° 37'$
19 : 00	$37° 57'$	$269° 18'$		$36° 29'$	$270° 23'$
19 : 15	$42° 8'$	$277° 5'$		$40° 34'$	$277° 51'$
19 : 30	$46° 7'$	$288° 8'$		$44° 29'$	$288° 19'$
19 : 45	$49° 21'$	$306° 39'$		$47° 45'$	$305° 31'$
19 : 54 : 40	$48° 13'$	$339° 53'$		$47° 4'$	$339° 33'$

9.14 A Note on Time Steps

So far in this whole chapter, the prediction of the path of the solar eclipse was made more difficult because time step we were working with was too large.

If we take small enough time steps, which is not difficult to do on a computer that automatically calculates the positions of the Sun and Moon at any arbitrary time, the calculation of all the extreme points of the various curves is rendered unnecessary. For example, in example 9.6, the extreme points of the rising / setting limits are simply the first and last points that produce possible values of ϵ. In example 9.7 and 9.10, the extreme points of the curves of maximum eclipse on the horizon and the curves of the limits of partial eclipse are determined by the first and last points $m \sin(M - Q)/p$ produce possible values of ψ, and the first and last points Δ is less than $|l|$. In example 9.9, the central contacts are the first and last points that produce real values of ζ_1.

Thus, if we take time steps of for example 1 second, not only do we have more accurate derivatives, we also, if we simply ignore times that give impossible solutions for points, automatically have the times of the various contacts and extreme points of curves accurate to the nearest second as those points are the first and last times that give possible solutions. The time of greatest eclipse can also be found to the nearest second by simply computing $m = \sqrt{x^2 + y^2}$ at each time and seeing when it is at minimum.

Therefore, the method of calculation of solar eclipse phenomena when taking small time steps is as follows:

1. Find a specific New Moon that produces a solar eclipse.

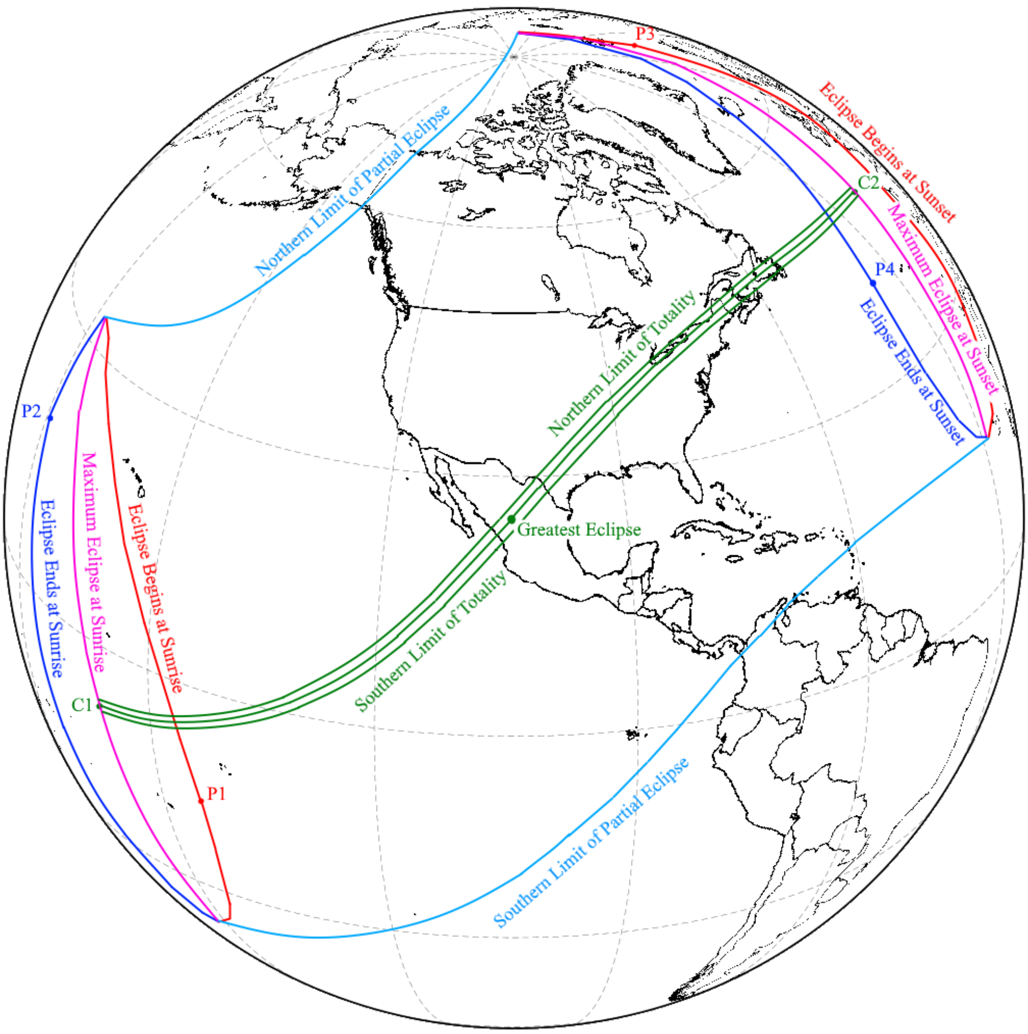
2. Calculate the Besselian elements at the time of New Moon.

3. Using this, calculate the first and last external penumbral contacts. This is necessary because we need the boundary times of the whole eclipse.
4. Now, beginning from the first external penumbral contact, calculate the elements at that time and calculate all the curves via methods detailed previously in this chapter, simply ignoring impossible solutions.
5. Increment by a small time step, recalculate the elements, and calculate new points until the last external penumbral contact is reached.

9.15 Eclipse Maps

The points we have computed in examples 9.5 through 9.11 can be plotted on a map and curves can be drawn through them:

The Total Solar Eclipse of April 8, 2024



Greatest Eclipse at 18:17 UTC Gamma = 0.3431
Duration of Totality at Greatest Eclipse = 4m 30.2s

9.16 Local Predictions

To predict the contacts for a location on the Earth with latitude ϕ and longitude λ, first calculate the place's fundamental coordinates at time T_0 using equation 9.6. Then, saying the contacts occurred at time $T_0 + \tau$, we can estimate x, y, ξ, and η as linear and write:

$$x = x_0 + x'\tau \qquad \xi = \xi_0 + \xi'\tau$$
$$y = y_0 + y'\tau \qquad \eta = \eta_0 + \eta'\tau$$

Where ξ' and η' are calculated with equation 9.40.
Then, the equations 9.18 become:

$$(l - i\zeta)\sin(Q) = x_0 + x'\tau - (\xi_0 + \xi'\tau)$$
$$(l - i\zeta)\cos(Q) = y_0 + y'\tau - (\eta_0 + \eta'\tau)$$

Which can be rearranged to:

$$(l - i\zeta)\sin(Q) = x_0 - \xi_0 + (x' - \xi')\tau$$
$$(l - i\zeta)\cos(Q) = y_0 - \eta_0 + (y' - \eta')\tau$$

Now if we set:

$$\left.\begin{cases} m\sin(M) = x_0 - \xi_0 \\ m\cos(M) = y_0 - \eta_0 \end{cases} \qquad \begin{cases} n\sin(N) = x' - \xi' \\ n\cos(N) = y' - \eta' \end{cases}\right\} \tag{9.79}$$

We have:

$$(l - i\zeta)\sin(Q) = m\sin(M) + \tau n\sin(N)$$
$$(l - i\zeta)\cos(Q) = m\cos(M) + \tau n\cos(N)$$

If we subtract N from all angles and set $Q - N = \psi$, we have:

$$\left.\begin{aligned} (l - i\zeta)\sin(\psi) &= m\sin(M - N) \\ (l - i\zeta)\cos(\psi) &= m\cos(M - N) + \tau n \end{aligned}\right\} \tag{9.80}$$

And thus:

$$\left.\begin{aligned} \sin(\psi) &= \frac{m\sin(M - N)}{l - i\zeta} \\ \tau &= \frac{(l - i\zeta)\cos(\psi) - m\cos(M - N)}{n} \\ T &= T_0 + \tau \\ Q &= N + \psi \end{aligned}\right\} \tag{9.81}$$

Where Q is the position angle of the contact on the disc of the Sun.

Now, for the time of maximum eclipse, if we put Δ as the distance to the shadow at that time, we have (using the same variables as before):

$$\left.\begin{aligned} \Delta\sin(\psi) &= m\sin(M - N) \\ \Delta\cos(\psi) &= m\cos(M - N) + \tau n \end{aligned}\right\} \tag{9.82}$$

And if we add the square of both equations, we have:

$$\Delta^2 = m^2\sin^2(M - N) + (m\cos(M - N) + \tau n)^2 \tag{9.83}$$
$$= m^2\sin^2(M - N) + m^2\cos^2(M - N) + 2\tau nm\cos(M - N) + \tau^2 n^2$$
$$= m^2 + 2\tau nm\cos(M - N) + \tau^2 n^2 \tag{9.84}$$

In order for Δ^2 to be minimum, we take the derivative of equation 9.84 and set it to 0.

$$\frac{d\Delta^2}{d\tau} = 2nm\cos(M-N) + 2\tau n^2$$
$$0 = 2nm\cos(M-N) + 2\tau n^2$$
$$\therefore \tau = -\frac{m\cos(M-N)}{n} \tag{9.85}$$
$$T = T_0 + \tau \tag{9.86}$$

And substituting this value of τ into equation 9.83 we have:

$$\Delta^2 = m^2\sin^2(M-N)$$
$$\therefore \Delta = |m\sin(M-N)| \tag{9.87}$$

For the position angle at maximum eclipse, we can see that in the first equation of 9.82, we have:

$$|m\sin(M-N)\sin(\psi)| = m\sin(M-N)$$
$$\therefore \sin(\psi) = \pm 1$$
$$\therefore \psi = \pm\pi/2$$

Since $Q = N \pm \psi$, we have:

$$Q = N \pm \pi/2 \tag{9.88}$$

The sign must be chosen such to make the path of the Moon across the Sun's disk make sense.

Example 9.12 Make a first approximation for the solar eclipse of April 8, 2024 as seen from Albuquerque, USA. ($\phi = 35°$, $\lambda = -107°$)

Solution
ϕ' and ρ at this latitude is calculated in example 7.1 to be 0.60771293 rad and 0.99890313 R_E.

We will use $T_0 = 18:00$. The standard sidereal time at this time was:

$$\Theta = 1.87597223 \text{ rad}$$

Thus the local sidereal time was:

$$\Theta_L = 1.87597223 \text{ rad} - 107° = 0.00846994 \text{ rad}$$

The equatorial coordinates of the observer are (by equation 9.6):

$$\begin{bmatrix} 0.99890313\cos(0.60771293)\cos(0.00846994) \\ 0.99890313\cos(0.60771293)\sin(0.00846994) \\ 0.99890313\sin(0.60771293) \end{bmatrix} = \begin{bmatrix} 0.82002616 \\ 0.00694573 \\ 0.57036507 \end{bmatrix}$$

The rotation matrix for this time was already calculated in example 9.2 as:

$$R = \begin{bmatrix} -0.302104542 & 0.9532748008 & 0 \\ -0.12376256 & -0.039221882 & 0.991536420 \\ 0.945206683 & 0.299547656 & 0.12982884 \end{bmatrix}$$

Thus the fundamental coordinates of the observer are:

$$\boldsymbol{\varrho} = \begin{bmatrix} -0.302104542 & 0.9532748008 & 0 \\ -0.12376256 & -0.039221882 & 0.991536420 \\ 0.945206683 & 0.299547656 & 0.12982884 \end{bmatrix} \begin{bmatrix} 0.82002616 \\ 0.00694573 \\ 0.57036507 \end{bmatrix} = \begin{bmatrix} -0.24111244 \\ 0.46377678 \\ 0.85122462 \end{bmatrix}$$

Now using $\mu' = 0.26187054$, $d = 0.13019636$, and $d' = 0.00025898$ (see tables in section 9.18), by equation 9.40:

$$\begin{aligned}
\xi' &= 0.26187054(-0.46377678\sin(0.13019636) + 0.85122462\cos(0.13019636)) \\
&= 0.20525638 \\
\eta' &= 0.26187054(-0.24111244)\sin(0.13019636) - 0.00025898 \cdot 0.85122462 \\
&= -0.00841788
\end{aligned}$$

Now, by equations 9.79, using $x_0 = -0.30856088$, $y_0 = 0.22479055$, $x' = 0.51147366$, and $y' = 0.27129112$:

$$\begin{aligned}
m\sin(M) &= -0.30856088 - (-0.24111244) &&= -0.06744844 \\
m\cos(M) &= 0.22479055 - 0.46377678 &&= -0.23898623 \\
\therefore m &= \sqrt{(-0.06744844)^2 + (-0.23898623)^2} &&= 0.24832179 \\
\therefore M &= \arctan(-0.06744844, -0.23898623) &&= -2.86651977 \text{ rad} \\
n\sin(N) &= 0.51147366 - 0.20525638 &&= 0.30621728 \\
n\cos(N) &= 0.27129112 - (-0.00841788) &&= 0.279709 \\
\therefore n &= \sqrt{(0.30621728)^2 + (0.279709)^2} &&= 0.41473624 \\
\therefore N &= \arctan(0.30621728, 0.279709) &&= 0.830609 \text{ rad}
\end{aligned}$$

Then, by equations 9.81, using $l_1 = 0.53573027$ and $i_1 = 0.00466835$:

$$\begin{aligned}
l - i\zeta &= 0.53573027 - 0.00466835 \cdot 0.85122462 = 0.53175646 \\
\sin(\psi) &= \frac{0.24832179\sin(-2.86651977 - 0.830609)}{0.53175646} \\
&= 0.24628686 \\
\therefore \psi_1 &= \pi - \arcsin(0.24628686) = 2.89274543 \\
\tau_1 &= \frac{0.53175646\cos(2.89274543) - 0.24832179\cos(-2.86651977 - 0.830609)}{0.41473624} \\
&= -0.73395597 \\
T_1 &= 18:00:00 - 0.73395596h = 17:15:58 \\
Q_1 &= 0.830609 + 2.89274543 = 3.72335443 = 213^\circ
\end{aligned}$$

$$\begin{aligned}
\therefore \psi_4 &= \arcsin(0.24628686) = 0.24884723 \\
\tau_4 &= \frac{0.53175646\cos(0.24884723) - 0.24832179\cos(-2.86651977 - 0.830609)}{0.41473624} \\
&= 1.75136701 \\
T_4 &= 18:00:00 + 1.75136701h = 19:45:05 \\
Q_4 &= 0.830609 + 0.24884723 = 1.07945623 = 62^\circ
\end{aligned}$$

For maximum eclipse, we use equation 9.85 and 9.86 for the time:

$$\begin{aligned}
\tau &= -\frac{0.24832179\cos(-2.86651977 - 0.830609)}{0.41473624} = 0.50870552 \\
T &= 18:00:00 + 0.50870552h = 18:30:31
\end{aligned}$$

and for Δ we use equation 9.87:

$$\Delta = |0.24832179\sin(-2.86651977 - 0.830609)| = 0.13096463$$

The magnitude then is given by equation 9.70. We use $l_2 = -0.01038565$ and $i_2 = 0.0046451$ and obtain:

$$L_1 = 0.53175646$$
$$L_2 = -0.01038565 - 0.0046451 \cdot 0.85122462 = -0.01433967$$
$$\therefore \text{Magnitude} = \frac{0.53175646 - 0.13096463}{0.53175646 + (-0.01433967)} = 0.77460152$$

The position angle of the Moon at this time is found by equation 9.88:

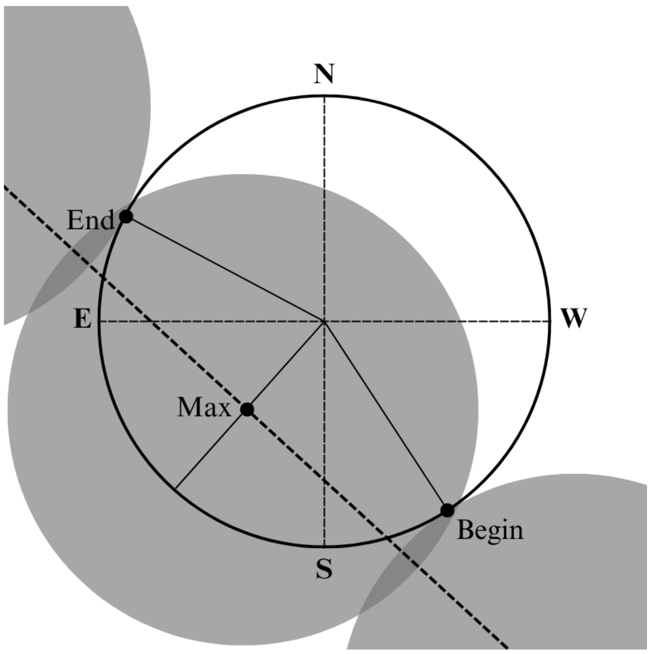
$$Q = 0.830609 + \pi/2 = 2.40140533 = 138^\circ$$

Our results may be arranged into a chart:

Solar Eclipse of April 8, 2024 – Predictions for Albuquerque, USA

Event	Time	Pos. Angle	Magnitude
Partial Eclipse Begins	17 : 15 : 58	213°	0%
Maximum Eclipse	18 : 30 : 31	138°	77%
Partial Eclipse Ends	19 : 45 : 05	62°	0%

Or a diagram:



For more accurate predictions, we must make more approximations just like we did with global contacts in example 9.5.

9.17 Lunar Occultations

For lunar occultations, simply replace the Sun with the star or planet. For the special case of the lunar occultations of stars, because stars are infinitely far away and have zero parallax and zero angular diameter, star rays can be modeled as parallel and therefore the shadow cone becomes a shadow cylinder: $f = 0$, and so $i = \tan(f) = 0$, and so $l = L =$ Radius of the Moon (the penumbra will not exist, the star will either be fully obstructed or not), and the coordinates z and ζ will not need to be used. The position of the Sun must also be considered as it is generally useless to predict lunar occultations of stars during daylight.

9.18 Table of Besselian Elements

Elements are calculated by the method of example 9.2. All angles are given in radians and all distances are given in Earth equatorial radii. All values are rounded to eight decimal digits, then trailing zeros are dropped.

Besselian Elements

t	d	μ	x	y	i_1	i_2	l_1	l_2
15 : 30	0.12954747	0.91439489	−1.58705613	−0.4537738	0.00466847	0.00464522	0.53549659	−0.01061816
15 : 45	0.12961246	0.97986288	−1.45925222	−0.38588218	0.00466846	0.00464521	0.53552714	−0.01058777
16 : 00	0.12967744	1.04533016	−1.3314264	−0.31802844	0.00466845	0.0046452	0.53555609	−0.01055896
16 : 15	0.12974243	1.11079815	−1.2036203	−0.2501576	0.00466844	0.00464519	0.53558345	−0.01053174
16 : 30	0.12980741	1.17626615	−1.07575353	−0.18229721	0.00466842	0.00464517	0.53560921	−0.0105061
16 : 45	0.12987191	1.24173342	−0.9479087	−0.11441956	0.00466841	0.00464516	0.53563338	−0.01048206
17 : 00	0.1299369	1.30720141	−0.82004606	−0.04657948	0.0046684	0.00464515	0.53565595	−0.0104596
17 : 15	0.13000188	1.37266941	−0.69216665	0.02127792	0.00466838	0.00464513	0.53567692	−0.01043873
17 : 30	0.13006687	1.43813669	−0.56435312	0.08909802	0.00466837	0.00464512	0.5356963	−0.01041945
17 : 45	0.13013186	1.50360469	−0.4364434	0.15693548	0.00466836	0.00464511	0.53571408	−0.01040175
18 : 00	0.13019636	1.56907269	−0.30856088	0.22479055	0.00466835	0.0046451	0.53573027	−0.01038565
18 : 15	0.13026134	1.63453996	−0.18070657	0.29258104	0.00466833	0.00464508	0.53574486	−0.01037113
18 : 30	0.13032633	1.70000795	−0.05280003	0.36038923	0.00466832	0.00464507	0.53575786	−0.0103582
18 : 45	0.13039132	1.76547595	0.07511692	0.42818782	0.00466831	0.00464506	0.53576926	−0.01034685
19 : 00	0.13045582	1.83094323	0.20296178	0.49600427	0.00466829	0.00464504	0.53577907	−0.01033709
19 : 15	0.13052081	1.89641123	0.33089643	0.56378388	0.00466828	0.00464503	0.53578728	−0.01032892
19 : 30	0.1305858	1.96187923	0.45879763	0.6315541	0.00466826	0.00464502	0.5357939	−0.01032233
19 : 45	0.13065079	2.0273465	0.58666434	0.69931496	0.00466825	0.004645	0.53579893	−0.01031733
20 : 00	0.13071529	2.0928145	0.71457695	0.76709409	0.00466824	0.00464499	0.53580236	−0.01031392
20 : 15	0.13078028	2.1582825	0.842453	0.83483658	0.00466822	0.00464497	0.5358042	−0.01031209
20 : 30	0.13084527	2.22374978	0.97033214	0.90257	0.00466821	0.00464496	0.53580445	−0.01031184
20 : 45	0.13090978	2.28921778	1.09821341	0.97029443	0.0046682	0.00464495	0.5358031	−0.01031318
21 : 00	0.13097477	2.35468505	1.22605493	1.03800983	0.00466818	0.00464493	0.53580017	−0.0103161

Right ascension, declination, and distance data were taken from the JPL Horizons System at *https://ssd.jpl.nasa.gov/horizons/*.

Derivatives of Besselian Elements

t	d'	μ'	x'	y'	l_1'	l_2'
15 : 30	0.00025996	0.26187196	0.51121564	0.27156648	0.0001222	0.00012156
15 : 45	0.00025994	0.26187054	0.51125946	0.27149072	0.000119	0.0001184
16 : 00	0.00025994	0.26187054	0.51126384	0.27144916	0.00011262	0.00011206
16 : 15	0.00025994	0.26187198	0.51134574	0.27146246	0.00010624	0.00010572
16 : 30	0.00025896	0.26187054	0.5114232	0.27147608	0.00009986	0.00009936
16 : 45	0.00025898	0.26187052	0.51141494	0.27143546	0.00009348	0.000093
17 : 00	0.00025994	0.26187198	0.5114841	0.27139496	0.00008708	0.00008666
17 : 15	0.00025994	0.26187056	0.51138588	0.271355	0.0000807	0.0000803
17 : 30	0.00025996	0.26187056	0.5114465	0.27131512	0.00007432	0.00007396
17 : 45	0.00025898	0.261872	0.51158448	0.27138506	0.00006794	0.0000676
18 : 00	0.00025898	0.26187054	0.51147366	0.27129112	0.00006156	0.00006124
18 : 15	0.00025994	0.26187052	0.5115217	0.27119736	0.00005518	0.0000549
18 : 30	0.00025996	0.26187198	0.51164698	0.27121356	0.0000488	0.00004856
18 : 45	0.00025898	0.26187056	0.51152362	0.27123008	0.00004242	0.00004222
19 : 00	0.00025898	0.26187056	0.51155902	0.27119212	0.00003604	0.00003586
19 : 15	0.00025996	0.261872	0.5116717	0.27109966	0.00002966	0.00002952
19 : 30	0.00025996	0.26187054	0.51153582	0.27106216	0.0000233	0.00002318
19 : 45	0.00025898	0.26187054	0.51155864	0.27107998	0.00001692	0.00001682
20 : 00	0.00025898	0.261872	0.51157732	0.27104324	0.00001054	0.00001048
20 : 15	0.00025996	0.26187056	0.51151038	0.27095182	0.00000418	0.00000416
20 : 30	0.000259	0.26187056	0.51152082	0.2709157	−0.0000022	−0.00000218
20 : 45	0.000259	0.26187054	0.51144558	0.27087966	−0.00000856	−0.00000852
21 : 00	0.00025996	0.26186908	0.51136608	0.2708616	−0.00001172	−0.00001168

di/dt can be regarded as 0.
Derivatives were calculated as:

$$f'(t) = \frac{f(t + 0.25h) - f(t - 0.25h)}{0.5h}$$

However, the derivatives at the first and last entries were calculated as:

$$f'(t) = \frac{f(t + 0.25h) - f(t)}{0.25h} \qquad f'(t) = \frac{f(t) - f(t - 0.25h)}{0.25h}$$

Using the first equation for 15 : 30 and the second for 21 : 00.

A more sophisticated method is possible:

$$f'(t) = \frac{-f(t + 0.5h) + 8f(t + 0.25h) - 8f(t - 0.25h) + f(t + 0.5h)}{3h}$$

But was not used. See finite difference coefficients.

Geodetic Elements

t	ρ_1	d_1	ρ_2	d_2
15 : 30	0.99670325	0.1299784	0.99994414	0.12911795
15 : 45	0.99670331	0.13004359	0.99994408	0.12918272
16 : 00	0.99670336	0.13010879	0.99994403	0.1292475
16 : 15	0.99670342	0.13017398	0.99994397	0.12931227
16 : 30	0.99670348	0.13023918	0.99994391	0.12937704
16 : 45	0.99670353	0.13030389	0.99994386	0.12944133
17 : 00	0.99670359	0.13036909	0.9999438	0.12950611
17 : 15	0.99670364	0.13043428	0.99994375	0.12957088
17 : 30	0.9967037	0.13049948	0.99994369	0.12963566
17 : 45	0.99670376	0.13056468	0.99994364	0.12970043
18 : 00	0.99670381	0.13062939	0.99994358	0.12976473
18 : 15	0.99670387	0.13069459	0.99994352	0.1298295
18 : 30	0.99670392	0.13075979	0.99994347	0.12989428
18 : 45	0.99670398	0.13082499	0.99994341	0.12995906
19 : 00	0.99670404	0.1308897	0.99994336	0.13002335
19 : 15	0.99670409	0.1309549	0.9999433	0.13008813
19 : 30	0.99670415	0.1310201	0.99994324	0.13015291
19 : 45	0.99670421	0.1310853	0.99994319	0.13021769
20 : 00	0.99670426	0.13115001	0.99994313	0.13028198
20 : 15	0.99670432	0.13121521	0.99994308	0.13034676
20 : 30	0.99670437	0.13128041	0.99994302	0.13041154
20 : 45	0.99670443	0.13134513	0.99994296	0.13047584
21 : 00	0.99670449	0.13141033	0.99994291	0.13054062

Auxiliary Elements

t	a'_1	b'_1	c'_1	a'_2	b'_2	c'_2
15 : 30	−0.00204673	−0.325256	0.49651364	−0.00203651	−0.32525666	0.49585165
15 : 45	−0.00188849	−0.32088099	0.49884782	−0.00187907	−0.32088165	0.49818584
16 : 00	−0.00172705	−0.31653541	0.50114349	−0.00171845	−0.31653607	0.50048151
16 : 15	−0.00156564	−0.31224127	0.5035195	−0.00155785	−0.31224193	0.50285753
16 : 30	−0.00140415	−0.30794063	0.50589309	−0.00139715	−0.30794129	0.50523113
16 : 45	−0.0012427	−0.30358228	0.50818381	−0.00123649	−0.30358294	0.50752186
17 : 00	−0.00108121	−0.29921943	0.51055294	−0.00107584	−0.29922009	0.50989099
17 : 15	−0.00091972	−0.29485192	0.51275756	−0.00091514	−0.29485258	0.51209563
17 : 30	−0.00075831	−0.29048257	0.51512205	−0.00075454	−0.29048323	0.51446012
17 : 45	−0.0005968	−0.28621554	0.51756681	−0.00059382	−0.2862162	0.51690488
18 : 00	−0.00043531	−0.28178103	0.51976556	−0.00043313	−0.28178169	0.51910364
18 : 15	−0.00027387	−0.27734348	0.52212331	−0.0002725	−0.27734414	0.5214614
18 : 30	−0.00011236	−0.27300982	0.52456126	−0.00011181	−0.27301048	0.52389936
18 : 45	0.00004915	−0.26867178	0.52675239	0.00004889	−0.26867244	0.5260905
19 : 00	0.00021057	−0.26427743	0.52910518	0.00020952	−0.26427809	0.5284433
19 : 15	0.0003721	−0.25982114	0.53153645	0.00037024	−0.2598218	0.53087457
19 : 30	0.00053356	−0.25541676	0.5337209	0.00053091	−0.25541742	0.53305903
19 : 45	0.000695	−0.25106449	0.53606613	0.00069155	−0.25106515	0.53540426
20 : 00	0.00085649	−0.24665174	0.53841018	0.00085223	−0.2466524	0.53774832
20 : 15	0.00101791	−0.24218143	0.54066943	0.00101284	−0.24218209	0.54000758
20 : 30	0.00117935	−0.23776187	0.54300819	0.00117347	−0.23776253	0.54234635
20 : 45	0.00134077	−0.23333814	0.54526312	0.0013341	−0.2333388	0.54460128
21 : 00	0.00149894	−0.22892955	0.54751567	0.00149149	−0.22893021	0.54685385

a_1', b_1', and c_1' are calculated with l_1, and a_2', b_2', and c_2' are calculated with l_2. Equation 9.49 was used.

Note that it is perfectly fine to interpolate the Besselian elements, their derivatives, and the geodetic elements for times not in the tables, it is best practice to recalculate these auxiliary elements for the intermediate times using those interpolated elements, instead of interpolating the auxiliary elements directly.

A Note on Interpolation

If one were to interpolate between the times given in the tables, a linear interpolation is enough in reasonable cases:

$$f(t) = \frac{f(t_1)(t_2 - t) + f(t_2)(t_1 - t)}{t_2 - t_1}$$

Where t_2 and t_1 are the times of two adjacent entries given by the table with $t_2 > t_1$. In this specific case, $t_2 - t_1$ is always $0.25h$ since the tables are given for every $0.25h$. However, one might notice that one cannot simply interpolate both the derivative of a function and the function itself linearly: if the derivative is linear then the function must be quadratic. However, because Besselian elements are so linear in nature (for most reasonable cases), linearly interpolating both the element and its derivative is excusable. However, if mathematical rigor is desired, when two points (t_1, y_1) and (t_2, y_2) and the derivatives at those points $f'(t_1) = d_1$ and $f'(t_2) = d_2$ are given: a cubic interpolation is needed:

$$f(t) = at^3 + bt^2 + ct + d$$

Where

$$a = \frac{1}{(t_1 - t_2)^2}\left(-\frac{2y_1}{t_1 - t_2} + \frac{2y_2}{t_1 - t_2} + d_1 + d_2\right)$$

$$b = \frac{1}{(t_1 - t_2)^2}\left(\frac{3(t_1 + t_2)y_1}{t_1 - t_2} - \frac{3(t_1 + t_2)y_2}{t_1 - t_2} - (t_1 + 2t_2)d_1 - (2t_1 + t_2)d_2\right)$$

$$c = \frac{1}{(t_1 - t_2)^2}\left(-\frac{6t_1t_2y_1}{t_1 - t_2} + \frac{6t_1t_2y_2}{t_1 - t_2} + \left(t_2^2 + 2t_1t_2\right)d_1 + \left(t_1^2 + 2t_2t_1\right)d_2\right)$$

$$d = \frac{1}{(t_1 - t_2)^2}\left(\frac{\left(3t_1t_2^2 - t_2^3\right)y_1}{t_1 - t_2} + \frac{\left(t_1^3 - 3t_1^2t_2\right)y_2}{t_1 - t_2} - t_1t_2^2d_1 - t_1^2t_2d_2\right)$$

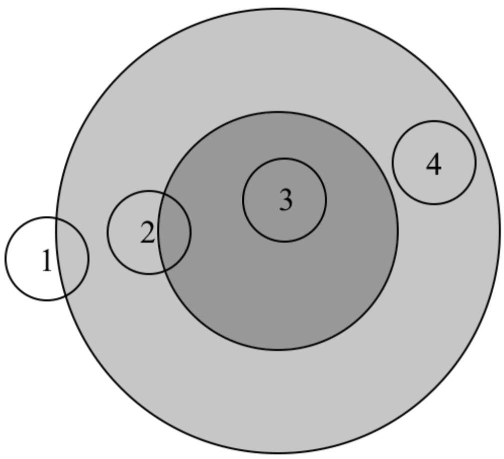
Chapter 10

Lunar Eclipses

Lunar eclipses happen when the Earth casts a shadow on the Moon. As it is a phenomenon that really changes the appearance of the Moon, anywhere on the Earth that can see the Moon during a lunar eclipse will see it, unlike solar eclipses. This makes the discussion of lunar eclipses much simpler than the discussion of solar eclipses. We follow Chauvenet (1891) and Seidelmann (1992).

10.1 Penumbral, Partial, and Total Lunar Eclipses

The Earth has two shadows: the penumbra and umbra. Depending on which parts of the shadow the Moon passes through, the eclipse can be categorized in four ways:



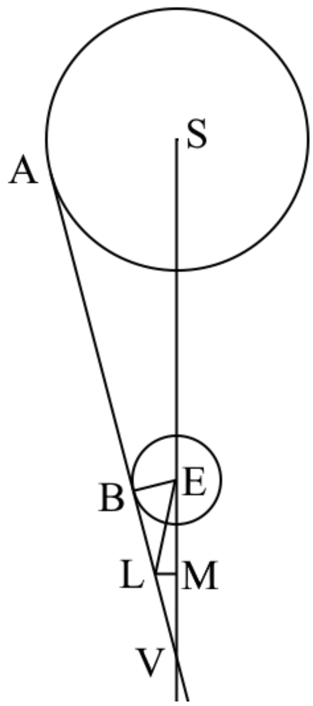
In this diagram, the penumbra of the Earth is the lightly shaded region (the larger shaded circle) and the umbra is the heavily shaded region (the smaller shaded circle). Depending on the position of the Moon, (1, 2, 3, or 4), the eclipse can be categorized as:

1. Partial Penumbral Eclipse
2. Partial (Umbral) Eclipse
3. Total (Umbral) Eclipse
4. Total Penumbral Eclipse

However, because penumbral eclipses are hardly noticeable, both 1 and 4 are simply collectively referred to as "penumbral eclipses", and the distinction between partial and total penumbral eclipse is usually not made.

10.2 Conditions for Eclipse

While a lunar eclipse can be modeled as a solar eclipse from the Moon, it is easier to simply consider the angular separation between the Moon and the shadow of the Earth to compute a lunar eclipse. For all intents and purposes, we can say that the shadow of the Earth is a circlular disk located at the antipode of the Sun, projected at the distance of the Moon, and thus the right ascension and declination of the center of the shadow are simply $\alpha_{\text{Sun}} + 12^h$ and $-\delta_{\text{Sun}}$ respectively.



In this diagram, if S is the center of the Sun and E is the center of the Earth, and LM is the radius of the Earth's shadow at the distance of the Moon, then the apparent radius of the shadow of the Earth is LEM.

We further have that:

$$\begin{aligned} LEM &= BLE - LVE \\ &= BLE - (AES - EAV) \\ &= \pi - s' + \pi' \end{aligned}$$

Where π is the equatorial horizontal parallax of the Moon, s' is the apparent radius of the Sun, and π' is the equatorial horizontal parallax of the Sun.
The above is the case for total lunar eclipse. For penumbral lunar eclipses, the apparent radius of the Earth's penumbra is given as $\pi + s' + \pi'$.

Now putting the least angular separation between the Moon and the shadow of the Earth as Σ, and the angular radius of the Earth's penumbra as f_1 and the angular radius of the umbra as f_2, we have:

$$\left.\begin{aligned} f_1 &= \pi + s' + \pi' \\ f_2 &= \pi - s' + \pi' \end{aligned}\right\} \quad (10.1)$$

And if we define these variables, putting s as the apparent radius of the Moon:

$$\left.\begin{aligned} L_1 &= f_1 + s \\ L_2 &= f_1 - s \\ L_3 &= f_2 + s \\ L_4 &= f_2 - s \end{aligned}\right\} \quad (10.2)$$

We have:

$$\left.\begin{aligned} \text{Condition for Partial Penumbral Eclipse: } & \Sigma < L_1 \\ \text{Condition for Total Penumbral Eclipse: } & \Sigma < L_2 \\ \text{Condition for Partial Eclipse: } & \Sigma < L_3 \\ \text{Condition for Total Eclipse: } & \Sigma < L_4 \end{aligned}\right\} \quad (10.3)$$

Where Σ is given by equation 9.1.

Note: due to atmospheric effects, f_1 and f_2 usually appear bigger than they truly are. They appear bigger by a factor of about 1.02 for our Earth, but for simplicity I will ignore this effect here.

Example 10.1 Determine if the full Moon of September, 2024 will result in a lunar eclipse.

Solution
We can find the time of opposition via the method of example 4.4, but by solving for when the elongation is 180°. We redefine the elongation to be expressed in the range of $[0°, 360°)$ to resolve the problem of the discontinuity, then subtract 180° from the elongation then use the method of bisection to solve for when this new value of elongation − 180° is 0°.

We find that the time of opposition was September 18, 2024, at 02 : 36. At this time:

$$\begin{aligned}
\lambda_{\text{Moon}} &= 355^\circ\ 20'\ 41.65'' \\
\lambda_{\text{Sun}} &= 175^\circ\ 20'\ 41.65'' \\
\beta_{\text{Moon}} &= -1^\circ\ 0'\ 15.19'' \\
\Delta_{\text{Moon}} &= 357\,484.36 \text{ km} \\
\Delta_{\text{Sun}} &= 150\,315\,226.07 \text{ km}
\end{aligned}$$

Also:

$$\begin{aligned}
\text{Earth's equatorial radius } &= 6378.137 \text{ km} \\
\text{Moon's radius } &= 1737.4 \text{ km} \\
\text{Sun's radius } &= 696\,000 \text{ km} \\
I &= 5.14^\circ
\end{aligned}$$

Thus by equations 4.1 and 8.6:

$$\begin{aligned}
s &= \arcsin\left(\frac{1737.4}{357\,484.36}\right) &&= 16'\ 42.5'' \\
s' &= \arcsin\left(\frac{696\,000}{150\,315\,226.07}\right) &&= 15'\ 55.07'' \\
\pi &= \arcsin\left(\frac{6378.137}{357\,484.36}\right) &&= 1^\circ\ 1'\ 20.32'' \\
\pi' &= \arcsin\left(\frac{6378.137}{150\,315\,226.07}\right) &&= 8.75''
\end{aligned}$$

Therefore, by equation 10.1 and 10.2:

$$\begin{aligned}
f_1 &= 1^\circ\ 1'\ 20.32'' + 15'\ 55.07'' + 8.75'' = 1^\circ\ 17'\ 24.13'' \\
f_2 &= 1^\circ\ 1'\ 20.32'' - 15'\ 55.07'' + 8.75'' = 0^\circ\ 45'\ 34''
\end{aligned}$$

$$\begin{aligned}
L_1 &= 1^\circ\ 17'\ 24.13'' + 16'\ 42.5'' &&= 1^\circ\ 34'\ 6.6'' \\
L_2 &= 1^\circ\ 17'\ 24.13'' - 16'\ 42.5'' &&= 1^\circ\ 0'\ 41.67'' \\
L_3 &= 0^\circ\ 45'\ 34'' + 16'\ 42.5'' &&= 1^\circ\ 2'\ 16.47'' \\
L_4 &= 0^\circ\ 45'\ 34'' - 16'\ 42.5'' &&= 0^\circ\ 28'\ 51.54''
\end{aligned}$$

To find Σ, we need q (equations 9.1 and 9.2), and therefore we need σ and μ, the derivatives of longitudes of the Sun and Moon. We take one hour as the time step. At 03 : 36:

$$\begin{aligned}
\lambda_{\text{Moon}} &= 355^\circ\ 58'\ 39.95'' \\
\lambda_{\text{Sun}} &= 175^\circ\ 22'\ 54.53''
\end{aligned}$$

Therefore:

$$\begin{aligned}
\mu &= \frac{355^\circ\ 58'\ 39.95'' - 355^\circ\ 20'\ 41.65''}{1h} = 37.97'/h \\
\sigma &= \frac{175^\circ\ 22'\ 54.53'' - 175^\circ\ 20'\ 41.65''}{1h} = 2.21'/h
\end{aligned}$$

Therefore, by equation 9.2:

$$\begin{aligned}
q &= \frac{37.97}{2.21} = 17.181 \\
\therefore I' &= \arctan\left(\frac{17.181}{17.181-1}\tan(5.14^\circ)\right) = 5.456^\circ
\end{aligned}$$

Therefore, by equation 9.1, Σ is:

$$\Sigma = -1^\circ\, 0'\, 15.19'' \cos(5.456^\circ) = -59'\, 58.81''$$

But, being a distance, Σ must be positive, so we take the absolute value. Comparing the value of Σ to the values of L, we can see that:

$$L_4 < \Sigma < L_3$$

And therefore, by equation 10.3, a partial eclipse of the Moon will occur.

10.3 Besselian Elements

If we have that:

$$\begin{aligned} v, u, w &= \text{The normalized geocentric equatorial cartesian coordinates of the Moon} \\ \alpha, \delta &= \text{The geocentric equatorial spherical coordinates of the Sun} \end{aligned}$$

Then the direction of the shadow is simply given by the opposite direction to the Sun: $a = \alpha + 180^\circ$, and $d = -\delta$. The coordinate transformation to the fundamental plane is identical to equation 9.5:

$$a = \alpha + 12^h \quad \text{and} \quad d = -\delta$$

$$R = \begin{bmatrix} -\sin(a) & \cos(a) & 0 \\ -\cos(a)\sin(d) & -\sin(a)\sin(d) & \cos(d) \\ \cos(a)\cos(d) & \sin(a)\cos(d) & \sin(d) \end{bmatrix} \tag{10.4}$$

And then the Moon's coordinates in the fundamental frame is given by:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = R \begin{bmatrix} v \\ u \\ w \end{bmatrix} \tag{10.5}$$

We use the normalized coordinates of the Moon because we are dealing with angular distances on the celestial sphere and not true distances. When the angles are expressed in radians, angular distance corresponds to arc length.

And thus we have the Besselian elements for lunar eclipses: x, y, x', y', s, L_1, L_2, L_3, and L_4, all expressed in radians.

Example 10.2 Determine the Besselian elements for the lunar eclipse of September 18, 2024 for the time 03 : 00.

Solution
Let r, α, δ be the equatorial spherical coordinates of the Moon and r', α', δ' be the equatorial spherical coordinates of the Sun. From an ephemeris, we find that on September 18, 2024 at 03 : 00:

$$\begin{aligned} r &= 357\,469.796 \text{ km} \\ \alpha &= 6.21947715 \\ \delta &= -0.0462551 \\ r' &= 150\,314\,543.493 \text{ km} \\ \alpha' &= 3.06722563 \\ \delta' &= 0.03220181 \end{aligned}$$

Where all angles are in radians. Thus, by equation 10.4:

$$a = 3.06722563 + \pi = 6.20881828$$
$$d = -0.03220181$$

And the rotation matrix is:

$$R = \begin{bmatrix} 0.0742985 & 0.99723605 & 0 \\ 0.03210726 & -0.00239213 & 0.99948157 \\ 0.99671905 & -0.07425998 & -0.03219624 \end{bmatrix}$$

The normalized cartesian coordinates of the Moon are:

$$(v, u, w) = (0.99690392, -0.06359697, -0.04623861)$$

And thus the Moon's coordinates in the fundamental frame is (by equation 10.5):

$$(x, y, z) = (0.01064727, -0.01405466, 0.99984454)$$

The units are radians because arc length = angle if the angle is expressed in radians. (The radius is omitted because we assume the celestial sphere to have radius 1.)

Since we need the derivatives x' and y' as well, we do the same calculations for the times $02:45$ and $03:15$ and obtain:

$$x_{02:45} = 0.00837454\,\text{rad}$$
$$y_{02:45} = -0.01530998\,\text{rad}$$
$$x_{03:15} = 0.01291948\,\text{rad}$$
$$y_{03:15} = -0.01279948\,\text{rad}$$

And therefore:

$$x' = \frac{0.01291948 - 0.00837454}{0.5h} = 0.00908988\,\text{rad}/h$$
$$y' = \frac{-0.01279948 - (-0.01530998)}{0.5h} = 0.005021\,\text{rad}/h$$

The remaining Besselian elements s and L are obtained in the same fashion as example 10.1:

$$s = 0.00486029\,\text{rad}$$
$$L_1 = 0.02737643\,\text{rad}$$
$$L_2 = 0.01765585\,\text{rad}$$
$$L_3 = 0.01811581\,\text{rad}$$
$$L_4 = 0.00839523\,\text{rad}$$

10.4 Contacts

We follow the same reasoning as in Chapter 9. Defining m as $m = \sqrt{x^2 + y^2}$, at the contacts, we have $m^2 = x^2 + y^2 = L^2$ (since $x^2 + y^2$ is the square of the distance between the center of the Moon and the center of the shadow), so if we assume x and y to change linearly, we have:

$$x = x_0 + x'\tau$$
$$y = y_0 + y'\tau$$

And so if we put:

$$\left.\begin{aligned} m_0\sin(M_0) &= x_0 \\ m_0\cos(M_0) &= y_0 \\ n\sin(N) &= x' \\ n\cos(N) &= y' \end{aligned}\right\} \quad (10.6)$$

We have:

$$\begin{aligned} L\sin(M) &= m_0\sin(M_0) + \tau n\sin(N) \\ L\cos(M) &= m_0\cos(M_0) + \tau n\cos(N) \end{aligned}$$

If we subtract N from all angles in the above equation, we get:

$$\begin{aligned} L\sin(M-N) &= m_0\sin(M_0-N) \\ L\cos(M-N) &= m_0\cos(M_0-N) + \tau n \end{aligned}$$

Now, if we put $\psi = M - N$, we have, for the time of contacts:

$$\left.\begin{aligned} \sin(\psi) &= \frac{m_0\sin(M_0-N)}{L} \\ \tau &= \frac{L\cos(\psi) - m_0\cos(M_0-N)}{n} \\ T &= T_0 + \tau \end{aligned}\right\} \quad (10.7)$$

The first equation has two solutions: we take $\psi = \arcsin()$ for last contacts and $\psi = 180° - \arcsin()$ for first contacts.

The first and last external contacts with the penumbra ($L = L_1$) are called P_1 and P_4, the first and last internal contacts with the penumbra ($L = L_2$) are called P_2 and P_3, the first and last external contacts with the umbra ($L = L_3$) are called U_1 and U_4, and the first and last internal contacts with the umbra ($L = L_4$) are called U_2 and U_3.

A visibility map can be drawn by determining at which points the Moon is visible at the contact times (i.e. determining for which points the Moon is above the horizon). This can be done by the method of example 6.6, optionally using the parallax corrected sunrise equation (equation 8.5) for extra accuracy.

Example 10.3 Make a first approximation of the contact times and eclipse durations of the lunar eclipse of September 18, 2024.

Solution
We will take $T_0 = 03:00$. Equations 10.6 give:

$$\begin{aligned} m_0\sin(M_0) &= 0.01064727 \\ m_0\cos(M_0) &= -0.01405466 \\ \therefore m_0 &= \sqrt{0.01064727^2 + (-0.01405466)^2} = 0.01763229 \\ \therefore M_0 &= \arctan(0.01064727, -0.01405466) = 2.49326968 \text{ rad} \\ n\sin(N) &= 0.00908988 \\ n\cos(N) &= 0.005021 \\ \therefore n &= \sqrt{0.00908988^2 + 0.005021^2} = 0.01038443 \\ \therefore N &= \arctan(0.00908988, 0.005021) = 1.06613342 \text{ rad} \end{aligned}$$

Now, for $L_1 = 0.02737643$ rad, equations 10.7 give:

$$\begin{aligned}
\sin(\psi) &= \frac{0.01763229 \sin(2.49326968 - 1.06613342)}{0.02737643} \\
&= 0.6374337 \\
\therefore \psi_{P1} &= \pi - \arcsin(0.6374337) = 2.45042966 \\
\tau_{P1} &= \frac{0.02737643 \cos(2.45042966) - 0.01763229 \cos(2.49326968 - 1.06613342)}{0.01038443} \\
&= -2.2743698 \\
T_{P1} &= 03:00:00 - 2.2743698h = 00:43:32
\end{aligned}$$

$$\begin{aligned}
\therefore \psi_{P4} &= \arcsin(0.6374337) = 0.69116299 \\
\tau_{P4} &= \frac{0.02737643 \cos(0.69116299) - 0.01763229 \cos(2.49326968 - 1.06613342)}{0.01038443} \\
&= 1.78818948 \\
T_{P4} &= 03:00:00 + 1.78818948h = 04:47:17
\end{aligned}$$

These are the times of beginning and end of the entire eclipse. Thus the whole eclipse lasts for $04:47:17 - 00:43:32 = 4h\ 3m$.

For the beginning and end of partial eclipse, we use $L = L_3 = 0.01811581$ and obtain:

$$\begin{aligned}
\sin(\psi) &= \frac{0.01763229 \sin(2.49326968 - 1.06613342)}{0.01811581} \\
&= 0.96328317 \\
\therefore \psi_{U1} &= \pi - \arcsin(0.96328317) = 1.84261887 \\
\tau_{U1} &= \frac{0.01811581 \cos(1.84261887) - 0.01763229 \cos(2.49326968 - 1.06613342)}{0.01038443} \\
&= -0.71147116 \\
T_{U1} &= 03:00:00 - 0.71147116h = 02:17:19
\end{aligned}$$

$$\begin{aligned}
\therefore \psi_{U4} &= \arcsin(0.96328317) = 1.29897378 \\
\tau_{U4} &= \frac{0.01811581 \cos(1.29897378) - 0.01763229 \cos(2.49326968 - 1.06613342)}{0.01038443} \\
&= 0.22529084 \\
T_{U4} &= 03:00:00 + 0.22529084h = 03:13:31
\end{aligned}$$

Thus partiality lasted for $03:13:31 - 02:17:19 = 56m$. Because this eclipse is partial, we do not have contacts with L_4 (the U_2 and U_3 contacts). Contacts with L_2 (the P_2 and P_3 contacts) are of little scientific importance.

Further approximations (just like the contact time calculations of chapter 9) are necessary for more accurate values.

10.5 Greatest Eclipse, Gamma, and Magnitude

The time of greatest eclipse is obtained exactly like in chapter 9. We must minimize $m^2 = x^2 + y^2$ and thus find the zero of the derivative

$$\frac{d(m^2)}{dt} = 2xx' + 2yy'$$

Since we can linearly approximate x and y as:

$$x = x_0 + x'\tau$$
$$y = y_0 + y'\tau$$

The derivative of $x^2 + y^2$ expands to:

$$\frac{d(m^2)}{dt} = 2(x_0x' + x'^2\tau) + 2(y_0y' + y'^2\tau)$$

Setting this to 0 and solving for τ gives:

$$\tau = -\frac{x_0x' + y_0y'}{x'^2 + y'^2} \tag{10.8}$$

And then T is given by $T = T_0 + \tau$. The gamma of the eclipse is defined by the true (not angular) distance from the center of the Moon to the axis of the shadow at the moment of greatest eclipse, and is given by multiplying m at the moment of greatest eclipse by the distance to the Moon (in units of Earth equatorial radii).

The magnitude of the eclipse at any time is given by:

$$\text{Magnitude } = \frac{L - m}{2s} \tag{10.9}$$

Where L can be L_1, in which case the magnitude is known as the *penumbral magnitude*, or it can be L_3, in which case the magnitude is known as the *umbral magnitude*.

Example 10.4 Make a first approximation of the time of greatest eclipse of the lunar eclipse of September 18, 2024.

Solution
We will take $T_0 = 03 : 00$. Equations 10.8 give:

$$\tau = -\frac{0.01064727 \cdot 0.00908988 + (-0.01405466) \cdot 0.005021}{0.00908988^2 + 0.005021^2}$$
$$= -0.24309017$$
$$\therefore T = 03 : 00 - 0.24309017h = 02 : 45 : 25$$

The location of greatest eclipse on the Earth is not easily definable, but it is generally given as the sublunar point at the time of greatest eclipse (see examples 6.10 and 8.5).

Example 10.5 Determine the penumbral and umbral magnitudes of the lunar eclipse of September 18, 2024 at time 03 : 00.

Solution
At 03 : 00, m is:

$$m = \sqrt{0.01064727^2 + (-0.01405466)^2} = 0.01763229$$

Using $L = L_1 = 0.02737643$ in equation 10.9, the penumbral magnitude is:

$$\text{Penumbral Magnitude } = \frac{0.02737643 - 0.01763229}{2 \cdot 0.00486029} = 1.0024$$

Using $L = L_3 = 0.01811581$ in equation 10.9, the umbral magnitude is:

$$\text{Umbral Magnitude } = \frac{0.01811581 - 0.01763229}{2 \cdot 0.00486029} = 0.0497$$

10.6 A Note on Time Steps

The same trick of small time steps for solar eclipses also works for lunar eclipses.

Chapter 11

Eclipse Patterns

This chapter will discuss when eclipses can occur and the patterns between them.

11.1 Eclipse Seasons

Because eclipses only occur when the Moon is near the nodes of the orbit of the Moon, the Sun must be also near the nodes for an eclipse to occur. For solar eclipses, the Sun must be at the same node as the Moon, and for lunar eclipses, the Sun must be at the opposite node as the Moon.

Recall the conditions for eclipse: equations 9.3, 10.2 and 10.3.

Eclipse Type	Condition
Partial Solar	$\beta < (s + s' + \pi - \pi') \sec(I')$
Central Solar	$\beta < (s - s' + \pi - \pi') \sec(I')$
Partial Penumbral Lunar	$\beta < (s + s' + \pi + \pi') \sec(I')$
Total Penumbral Lunar	$\beta < (-s + s' + \pi + \pi') \sec(I')$
Partial Lunar	$\beta < (s - s' + \pi + \pi') \sec(I')$
Total Lunar	$\beta < (-s - s' + \pi + \pi') \sec(I')$

The duration of time that the Sun is near enough to a lunar node for these conditions to be met at conjunction is known as an *eclipse season*. These eclipse seasons vary in length due to the fact that s, s', π, π', and I' are not constant values, but we can consider their average values, as in the following example.

Example 11.1 Using the average values of s, s', π, π', and I', Determine if two solar eclipses can occur one month apart.

Solution
Using:

$$\begin{aligned}
\text{Distance to the Moon} &= 384\,399 \text{ km} \\
\text{Distance to the Sun} &= 149\,598\,023 \text{ km} \\
\text{Radius of the Moon} &= 1737.4 \text{ km} \\
\text{Radius of the Sun} &= 696\,000 \text{ km} \\
\text{Equatorial Radius of the Earth} &= 6378.137 \text{ km} \\
\text{Orbital Period of the Moon} &= 27.32 \text{ dy} \\
\text{Orbital Period of the Earth} &= 365.25 \text{ dy} \\
\text{Lunar Orbital Inclination} &= 5.14^\circ
\end{aligned}$$

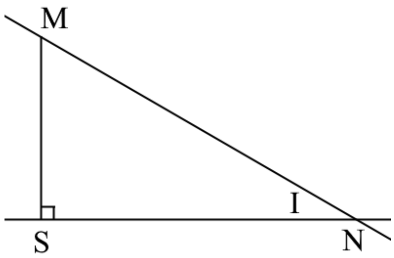
We have:

$$s = 0.259^\circ \qquad s' = 0.267^\circ$$
$$\pi = 0.951^\circ \qquad \pi' = 0.00244^\circ$$
$$\mu = 0.549^\circ/h \qquad \sigma = 0.0411^\circ/h$$
$$q = 13.358 \qquad I' = 5.553^\circ$$

Using all these values, we have for the conditions:

Eclipse Type	Condition
Partial Solar	$\beta < 1.482^\circ$
Central Solar	$\beta < 0.945^\circ$
Partial Penumbral Lunar	$\beta < 1.486^\circ$
Total Penumbral Lunar	$\beta < 0.966^\circ$
Partial Lunar	$\beta < 0.950^\circ$
Total Lunar	$\beta < 0.429^\circ$

Where only the absolute value of β matters.



Consider this diagram, showing the region near the lunar node N, where S and M are the positions of the Sun and Moon respectively at the moment of conjunction. The distance SM is β. It is evident that:

$$SN = \beta \cot(I) \qquad (11.1)$$

And therefore, for solar eclipses, since the maximum β is 1.482°, the maximum possible SN is $1.482^\circ \cot(5.14^\circ) = 16.476^\circ$, and since the Sun's longitude changes by $0.0411^\circ/h = 0.986^\circ/\text{dy}$, we have that it takes the Sun $16.476/0.986 = 16.71$ dy to traverse this distance.

Since the triangle can be mirrored on the other side, we have that there is a $16.71 \cdot 2 = 33.42$ dy period where solar eclipses are possible. Since the synodic month is only 29.53 dy long (see example 4.3), it is indeed possible for two solar eclipses to occur one month apart if one occurs at the beginning of the season and the other at the end of the season. Let us now discuss the nature of these eclipses.

If we do the same analysis with central solar eclipses, we have maximum $\beta = 0.945^\circ$ and thus $SN = 10.506^\circ$, and thus the Sun would take 10.655 dy to traverse that distance, and thus there is only a $10.655 \cdot 2 = 21.31$ dy period where central solar eclipses are possible. Since this is shorter than one month, two central eclipses of the Sun cannot occur one month apart.

If we consider the length of time 16.71 dy $+ 10.655$ dy $= 27.365$ dy, we see that this too is shorter than a synodic month, and therefore a central solar eclipse and a partial solar eclipse cannot occur one month apart.

If the eclipse season is very close to being one synodic month, a more precise approach is required because the nodal precession of the Moon's orbit cannot be ignored. The more precise method is shown in example 11.3.

Therefore we have that in order for two solar eclipses to occur one month apart, they must both be partial. This is exemplified by the partial solar eclipses of July 13, 2018 and August 11, 2018.

The eclipse season lengths are tabulated here for convenience.

Eclipse Type	Eclipse Season Length
Partial Solar	33.42 dy
Central Solar	21.31 dy
Partial Penumbral Lunar	33.51 dy
Total Penumbral Lunar	21.78 dy
Partial Lunar	21.42 dy
Total Lunar	9.68 dy

11.2 Eclipse Years and Draconic Months

Because the lunar nodes precess (see chapter 3 for more detail), it does not take the Sun and Moon one orbital period each to return to a node after leaving it. The time it takes for the Sun to return to the same node is known as an *eclipse year*, and the time it takes for the Moon to return to the same node is known as a *draconic month*. They are both calculable as:

$$T = \frac{OT_{\dot{\Omega}}}{T_{\dot{\Omega}} \pm O} \tag{11.2}$$

Where O is the orbital period of the Moon or the Sun, and $T_{\dot{\Omega}}$ is the nodal precession period of the Moon's orbit. The top sign is to be used if the node regresses, and the bottom sign if the node advances.

Example 11.2 Using $T_{\dot{\Omega}} = 6793\text{dy}$, Determine the length of the eclipse year and the draconic month.

Solution
Because the nodes regress for our Moon, we use the top sign in equation 11.2.

$$EY = \frac{365.25 \cdot 6793}{6793 + 365.25} = 346.61 \text{ dy}$$
$$DM = \frac{27.32 \cdot 6793}{6793 + 27.32} = 27.21 \text{ dy}$$

Because the Sun will make a whole circle relative to a lunar node in one eclipse year, we have that SN in the previous diagram, which we will now denote as ξ, will change by 360° every eclipse year.

Example 11.3 Determine if a central eclipse of the Sun can occur a fortnight after a total eclipse of the Moon.

Solution
In one fortnight, ξ will change by

$$\Delta\xi = \frac{360°}{346.61 \text{ dy}} \cdot \frac{29.53\text{dy}}{2} = 15.34°$$

From the conditions, we know that for a total lunar eclipse to occur, β must be at maximum 0.429°, and therefore ξ must be at maximum $0.429° \cot(5.14°) = 4.77°$. Similarly, ξ must be at maximum 10.51° for a central solar eclipse. If we assume a total lunar eclipse occurred when the Sun was 4.77° to the west of the node, after one fortnight the Sun would be at $-4.77° + 15.34° = 10.57°$ east of the node, slightly too far for a central solar eclipse to occur. However, with slightly different values of maximum β (remember, these numbers do not stay constant), it could well be possible. In fact, in 2032, there will be a total lunar eclipse on April 25, and an annular solar eclipse a fortnight later, on May 9.

Example 11.4 Determine the maximum number of eclipses that can occur in one year.

Solution
Because the solar eclipse season (33.42 dy) is longer than a synodic month, there could be two solar eclipses in one month. In between the two solar eclipses, there would be a lunar eclipse, for a total of three eclipses at maximum per eclipse season. (Similarly, there could also be two lunar eclipses and one solar eclipse in the middle).

Assume that at the beginning of the year, the Sun passed through a lunar node and an eclipse occurred. A fortnight later another eclipse would occur. Half an eclipse year later, the Sun would be at the opposite lunar node, and another eclipse season would start, with a maximum of three eclipses. Because an eclipse year is shorter than a year, The Sun would begin another eclipse season before the end of the year, which could produce three more eclipses, for a total of eight. However, we must count the eclipses carefully. Because an eclipse year is $346.61/29.53 = 11.74$ months long, half an eclipse year is 5.78 months long, which rounds up to 6 months (remember, eclipses only occur at new or full moon, meaning they can only occur separated by units of 0.5 months), and in this scenario we would have:

1. Eclipse at month 0
2. Eclipse at month 0.5
3. Eclipse at month 5.5
4. Eclipse at month 6
5. Eclipse at month 6.5
6. Eclipse at month 11.5
7. Eclipse at month 12
8. Eclipse at month 12.5 (!)

However, 12.5 months is $12.5 \cdot 29.53 = 369.13$ days, which is longer than a year, meaning there is no way for the eighth eclipse to fit in one year. Therefore the maximum number of eclipses that could occur is seven.

This is exemplified by the extraordinary year of 1982:

Date	Eclipse
January 9, 1982	Total Lunar Eclipse
January 25, 1982	Partial Solar Eclipse
June 21, 1982	Partial Solar Eclipse
July 6, 1982	Total Lunar Eclipse
July 20, 1982	Partial Solar Eclipse
December 15, 1982	Partial Solar Eclipse
December 30, 1982	Total Lunar Eclipse

11.3 Periodicity of Eclipses and the Saros

Because eclipses occur when the Moon and Sun are at nodes, if we can determine a length of time where the Sun and Moon will end up at a node again after an eclipse occurs, we can determine that an eclipse will also occur on that date. These lengths of time are known as *eclipse cycles*.

Example 11.5 Using the more accurate values of: Synodic Month $= 29.53059$ dy and Eclipse Year $= 346.62008$ dy, Determine the length of some eclipse cycles.

Solution
An eclipse is followed by an eclipse of the same type (solar or lunar) only in units of synodic months. Therefore, we need to find the number of synodic months that would result in the Sun being at a node. This ensures the parameter ξ is close enough for an eclipse to occur. Thus, we search for multiples of synodic months that are also multiples of eclipse years.

For example, 47 synodic months is $47 \cdot 29.53059 = 1387.93773$ days, which is $1387.93773/346.62008 = 4.0042$ eclipse years, which has an error of 0.0042 eclipse years off from a perfect multiple. This corresponds to a $\Delta\xi$ of $0.0042 \cdot 360° = 1.512°$. This means that every cycle, the distance of the Sun from a node changes only by 1.512°, meaning favorable conditions for an eclipse are reproduced.

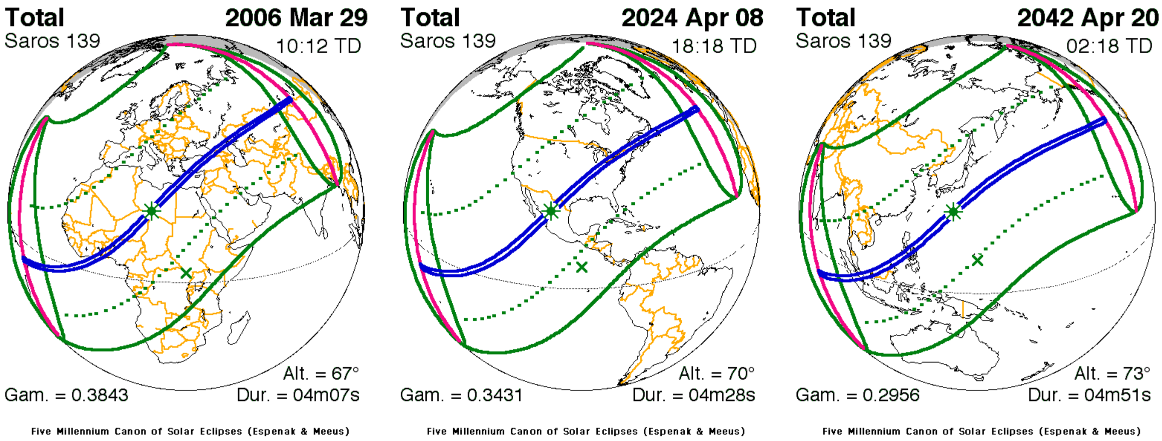
A list of periods with absolute $\Delta\xi$ less than 2° are listed here, computed via trial and error:

Syn. Months	Eclipse Years	Years	$\Delta\xi$
47	4.0042	3.8	1.5137°
176	14.9945	14.23	−1.9914°
223	18.9987	18.03	−0.4777°
270	23.0029	21.83	1.036°
446	37.9973	36.06	−0.9554°
493	42.0016	39.86	0.5583°
669	56.996	54.09	−1.4331°

One can see that the eclipse cycle with a period of 223 synodic months has a $\Delta\xi$ lower than the rest. This cycle is known as a *Saros* (plural *Saroi*) and is the standard eclipse cycle used today.

A Saros is special not only because it has a very small $\Delta\xi$ value per cycle, but because it is also an almost exact multiple of the anomalistic month (see chapter 3 for more details; one Saros is 238.992 anomalistic months), meaning the Moon is also at almost the same distance from the Earth after one Saros. Favorable conditions for the same kind of eclipse (total or annular) eclipse are reproduced. In addition, a Saros is also a near multiple of the solar year, having only 0.03 years (or eleven days) of error per cycle. This means that the Earth is also nearly at the same point in its orbit after one Saros, so the axis of the Earth is tilted in the same way. This means that eclipses separated by one Saros will not only be of similar type and duration, but also trace similar paths on the ground.

This is exemplified by this series of three solar eclipses, separated by one Saros each:



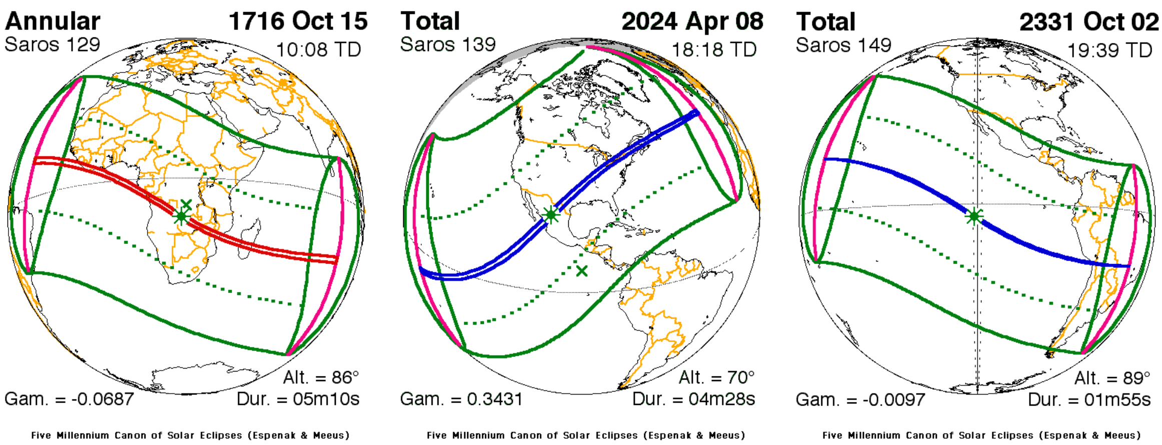
(From the *Five Millenium Catalog of Solar Eclipses* by Espenak and Meeus.)

Because a Saros is 223 months = 6585.324 days, the Earth rotates by 0.324 of a full turn (about 120°) between each eclipse in a Saros. This causes eclipses separated by one saros to occur in places separated by about 120° in longitude. A chain of eclipses separated by a Saros each is called a *Saros series*. The above shows three members in the solar Saros series number 139. There are many Saros series active at the same time at any given time.

For comparison, a cycle called an *Utting Cycle* exists, with an even smaller $\Delta\xi$ than a Saros:

Syn. Months	Eclipse Years	Anom. Months	Years	$\Delta\xi$
3803	323.9998	4075.7274	307.47	$-0.0749°$

However, because it is not a near multiple of an anomalistic month, and because it is not as close of a multiple of a year, the characteristics and paths of the eclipses separated by an Utting Cycle each are not similar.



(From the *Five Millenium Catalog of Solar Eclipses* by Espenak and Meeus.)

However, because two Utting cycles are $307.47 \cdot 2 = 614.94$ years, which is a near multiple of a year, every two Utting cycles there will be an eclipse with a similar shaped path. (However, because the anomalistic month still does not match, the duration and type of eclipse will not match.)

Despite the $\Delta\xi$ error for Saros cycles being very small, it still exists and therefore Saros series die out eventually.

Example 11.6 Determine the length and number of members in a solar Saros series.

Solution
We determined in example 11.1 that for a solar eclipse, the maximum ξ is 16.476° and for a central solar eclipse, the maximum ξ is 10.506°. Because each Saros period has a $\Delta\xi$ of −0.4777°, the number of members is:

$$\text{Num. Members } = \frac{2 \cdot 16.476}{|-0.4777|} + 2 = 71$$

And of these,

$$\text{Num. Central Members } = \frac{2 \cdot 10.506}{|-0.4777|} + 2 = 46$$

are central eclipses. This means that on average, we should expect solar saros series to consist of 12 (or 13) partial eclipses followed by 45 central eclipses followed by 13 (or 12) partial eclipses. This is a crude approximation however, as it ignores the variation in all variables, and every saros series is quite different from each other. (In reality, most Saros series have about 70 to 73 members (but can range from 69 to 87), and the number of central eclipses in each vary significantly, ranging from 39 to 59, with 43 being most common.)

In addition, since there are about 71 members in a Saros series, every Saros series lasts about

$$(71-1) \cdot 223 \text{ mo } \cdot 29.53059 \frac{\text{dy}}{\text{mo}} \cdot \frac{1 \text{ yr}}{365.25 \text{ dy}} = 1262 \text{ years}$$

In reality, Saros series take about 1250 to 1300 years from start to finish.

As an example of a full solar Saros series, Solar Saros 139 has 71 members, 7 partial eclipses followed by 55 central eclipses followed by 9 partial eclipses, and lasts for 1262 years.

As an example of a full lunar Saros series, Lunar Saros 118 has 73 members, 9 penumbral eclipses followed by 7 partial eclipses followed by 28 total eclipses followed by 8 partial eclipses followed by 21 penumbral eclipses, and lasts for 1298 years.

Chapter 12

Solar Transits

Transits of planets across the Sun are very similar to solar eclipses.

12.1 Conditions and Periodicity of Transits

One can use methods previously discussed in other chapters to analyze the conditions and periodicity of transits. However, because planets do not orbit the Earth, it is very difficult to calculate transit seasons and conditions from a geocentric perspective. A clever workaround is to model transits of planets across the Sun as eclipses of the Earth by planets as seen from the Sun.

Example 12.1 Given that for the most recent June 5/6, 2012 transit of Venus across the Sun, Venus was near its descending node and that $\beta_{\text{Venus}} = +0.157°$ at ecliptic conjunction, predict the dates of the next few Venus transits.

Solution
Analyzing transits of Venus across the Sun as eclipses of Earth by Venus as seen fom the sun, we will have s and s' as the apparent radii of Venus and the Earth as seen from the Sun, π and π' as the parallaxes of Venus and the Earth as seen from the Sun, and μ and σ as the mean motions in longitude of Venus and the Earth.
Assuming the orbits of the Earth and Venus are circular and using the average values given below:

$$\begin{aligned}
\text{Earth Sun Distance} &= 149\,598\,023 \text{ km} \\
\text{Venus Sun Distance} &= 108\,208\,927 \text{ km} \\
\text{Orbital Period of the Earth} &= 365.25 \text{ dy} \\
\text{Orbital Period of Venus} &= 224.7 \text{ dy} \\
\text{Radius of Venus} &= 6051.8 \text{ km} \\
\text{Radius of the Earth} &= 6378.137 \text{ km} \\
\text{Radius of the Sun} &= 696\,000 \text{ km} \\
\text{Venus Orbital Inclination} &= 3.39°
\end{aligned}$$

We have:

$$\begin{aligned}
s &= 0.0032° & s' &= 0.00244° \\
\pi &= 0.369° & \pi' &= 0.267° \\
\mu &= 0.0668°/h & \sigma &= 0.0411°/h \\
q &= 1.625 & I' &= 8.756°
\end{aligned}$$

Thus the condition for a transit is (by equation 9.3-i):

$$\beta_{\text{Heliocentric}} < 0.109°$$

Converting this to a condition for ξ, we have (by equation 11.1):

$$\xi_{\text{Heliocentric}} < 0.109^\circ \cot(3.39^\circ) = 1.84^\circ$$

Therefore Venus needs to be within 1.84° of a node in order for a transit to happen.

Now let us figure out what ξ was during the transit of 2012.
We have that $\beta = 0.157^\circ$, and that the distance from the Earth to Venus is

$$149\,598\,023 \text{ km} - 108\,208\,927 \text{ km} = 41\,389\,096 \text{ km}$$

so the geocentric z coordinate of Venus was

$$41\,389\,096 \text{ km} \cdot \sin(0.157^\circ) = 113\,413 \text{ km}$$

Now, since the distance to Venus from the Sun is 108 208 927 km, the heliocentric ecliptic latitude of Venus was:

$$\beta_{\text{Heliocentric}} = \arcsin\left(\frac{113\,413 \text{ km}}{108\,208\,927 \text{ km}}\right) = 0.06005^\circ$$

Converting this to ξ:

$$\xi_{\text{Heliocentric}} = 0.06005^\circ \cot(3.39^\circ) = 1.0138^\circ$$

But because Venus was yet to reach its descending node at conjunction (as can be inferred from the fact that its ecliptic latitude was positive during the transit), we should use $\xi = -1.0138^\circ$ instead.

Because the orbits of planets practically do not precess, the "eclipse year" for Venus transits is simply the regular solar year of 365.25 days. Because Venus is only in conjunction with the Sun every synodic period, which is 583.92 days for Venus (see example 4.3), we add multiples of 583.92 days.

After one synodic period, we have:

$$\xi_{\text{Heliocentric}} = -1.0138^\circ + \frac{360^\circ}{365.25 \text{ dy}} \cdot 583.92 \text{ dy} \cdot 1 = 574.513^\circ = 214.513^\circ$$

Which is not 1.84° within either the descending or ascending node. This table was computed in this brute force manner:

Syn. Cycles	$\xi_{\text{Heliocentric}}$	Transit Date	Node
71	1.396	2125/12	Ascending
76	−0.969	2133/12	Ascending
147	1.44	2247/06	Descending
152	−0.925	2255/06	Descending
223	1.485	2368/12	Ascending
228	−0.881	2376/12	Ascending
299	1.529	2490/06	Descending
304	−0.836	2498/06	Descending
375	1.573	2611/12	Ascending
380	−0.792	2619/12	Ascending
451	1.618	2733/06	Descending
456	−0.748	2741/06	Descending

For comparison, the actual table of future Venus transits is given below:

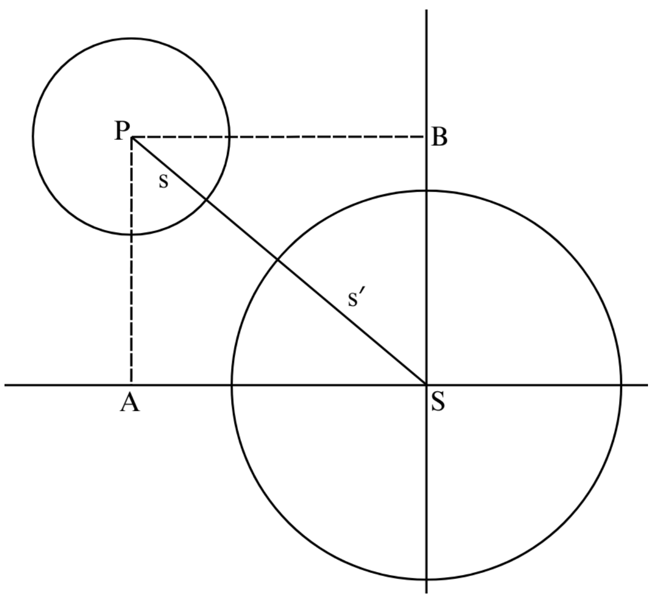
Syn. Cycles	Transit Date	Node
66	2117/12	Ascending
71	2125/12	Ascending
147	2247/06	Descending
152	2255/06	Descending
218	2360/12	Ascending
223	2368/12	Ascending
299	2490/06	Descending
304	2498/06	Descending
370	2603/12	Ascending
375	2611/12	Ascending
451	2733/06	Descending
456	2741/06	Descending

We can see that while the predictions for descending node transits are all correct, the ascending node transits are all wrong. This is because we assumed the Earth's orbit to be perfectly circular and used average values for everything in our calculations. In reality, instead of following a nice 71, 5, 71, 5 cycle between transits like we predicted, the pattern between Venus transits is 66, 5, 76, 5 cycle because the speed at which the Earth orbits the Sun is quite different in December compared to June.

This is clearly a very bad oversimplification, If a more precise prediction is wanted, one must analyze every conjunction separately as we did in sections 9.2 and sections 10.2.

12.2 Geocentric Calculations

While it is possible to calculate transits with the method detailed in chapter 9, it is possible to make quite a few abridgments. Let us calculate a transit of a planet across the Sun as seen from the center of the Earth.



In this diagram, let S be the position of the Sun, and P the position of the transiting planet at time T_0. Now let:

$$\alpha, \delta, s = \text{The right-ascension, declination, and apparent radius of the Planet}$$
$$\alpha', \delta', s' = \text{The right-ascension, declination, and apparent radius of the Sun}$$

Also, let the angular distance PS be denoted m and the angle BSP be denoted M. Then, we have:

$$m \sin(M) = PB$$
$$m \cos(M) = PA$$

Because the distances in this diagram are very small, we can approximate this diagram to be a plane and write that $PA = \delta - \delta'$.

However, for the distance PB, we cannot simply write $\alpha - \alpha'$ because the horizontal distance on a sphere (the celestial sphere) depends on the declination. In specific, the radius of the cross section of a sphere is proportional to cos(Declination). We can take the declination of the point in between B

and S as an approximation and write $PB = (\alpha - \alpha')\cos((\delta + \delta')/2)$.

$$\left.\begin{aligned} m\sin(M) &= (\alpha - \alpha')\cos(\delta_0) \\ m\cos(M) &= \delta - \delta' \end{aligned}\right\} \quad (12.1)$$

Where $\delta_0 = (\delta + \delta')/2$.

Now, letting

$$\begin{aligned} a = &\ \text{The relative motion in right ascension} = d\alpha/dt - d\alpha'/dt \\ d = &\ \text{The relative motion in declination} = d\delta/dt - d\delta'/dt \end{aligned}$$

If we say that the contact happened at time $T = T_0 + \tau$, We can express the difference in right ascension between the two bodies at the time of contact as $\alpha - \alpha' + a\tau$ and the difference in declination as $\delta - \delta' + d\tau$.

Also, at the contact times, evidently $m = s' + s$ for the external contacts and $m = s' - s$ for the internal contacts. If we say the value of M was Q at the time of contact, we can write:

$$\left.\begin{aligned} (s' \pm s)\sin(Q) &= (\alpha - \alpha' + a\tau)\cos(\delta_0) \\ &= (\alpha - \alpha')\cos(\delta_0) + a\cos(\delta_0)\tau \\ (s' \pm s)\cos(Q) &= \delta - \delta' + d\tau \end{aligned}\right\} \quad (12.2)$$

Now if we let:

$$\left.\begin{aligned} n\sin(N) &= a\cos(\delta_0) \\ n\cos(N) &= d \end{aligned}\right\} \quad (12.3)$$

Equation 12.2 can be rewritten as:

$$\begin{aligned} (s' \pm s)\sin(Q) &= m\sin(M) + \tau n\sin(N) \\ (s' \pm s)\cos(Q) &= m\cos(M) + \tau n\cos(N) \end{aligned}$$

From which if we subtract N from all angles and set $Q - N = \psi$, we obtain:

$$\left.\begin{aligned} \sin(\psi) &= \frac{m\sin(M - N)}{s' \pm s} \\ \tau &= \frac{(s' \pm s)\cos(\psi) - m\cos(M - N)}{n} \\ T &= T_0 + \tau \\ Q &= N + \psi \end{aligned}\right\} \quad (12.4)$$

Additionally, the time, position angle, and separation at greatest transit is given by equations 9.85 to 9.88:

$$\tau = -\frac{m\cos(M - N)}{n} \quad (9.85)$$

$$T = T_0 + \tau \quad (9.86)$$

$$\Delta = |m\sin(M - N)| \quad (9.87)$$

$$Q = N \pm \pi/2 \quad (9.88)$$

Example 12.2 Given that on June 6, 2012 at time $T_0 = 01:00$,

$$\begin{aligned} \alpha &= 1.29719560 \text{ rad} & \alpha' &= 1.29732941 \text{ rad} \\ \delta &= 0.39816972 \text{ rad} & \delta' &= 0.39542373 \text{ rad} \\ s &= 0.00014012 \text{ rad} & s' &= 0.00458491 \text{ rad} \end{aligned}$$

$$a = -0.00122246 \text{ rad}/h$$
$$d = -0.00029525 \text{ rad}/h$$

Make first approximations on the contact times and the time of greatest transit of the transit of Venus.

Solution
First, we find $\delta_0 = (0.39816972 + 0.39542373)/2 = 0.396796725$.
By equations 12.1 and 12.3:

$$\begin{aligned} m\sin(M) &= (1.29719560 - 1.29732941)\cos(0.396796725) &&= -0.00012341 \\ m\cos(M) &= 0.39816972 - 0.39542373 &&= 0.00274599 \\ \therefore m &= \sqrt{(-0.00012341)^2 + 0.00274599^2} &&= 0.00274876 \\ \therefore M &= \arctan(-0.00012341, 0.00274599) &&= -0.04491168 \text{ rad} \\ n\sin(N) &= -0.00122246\cos(0.396796725) &&= -0.00112748 \\ n\cos(N) &= -0.00029525 && \\ \therefore n &= \sqrt{(-0.00112748)^2 + (-0.00029525)^2} &&= 0.0011655 \\ \therefore N &= \arctan(-0.00112748, -0.00029525) &&= -1.8269125 \text{ rad} \end{aligned}$$

Now, for exterior contacts, equations 12.4 give:

$$\begin{aligned} \sin(\psi) &= \frac{0.00274876\sin(-0.04491168 - (-1.8269125))}{0.00458491 + 0.00014012} \\ &= 0.56881757 \\ \therefore \psi_1 &= \pi - \arcsin(0.56881757) = 2.53652518 \\ \tau_1 &= \frac{(0.00458491 + 0.00014012)\cos(2.53652518) - 0.00274876\cos(-0.04491168 - (-1.8269125))}{0.0011655} \\ &= -2.83991567 \\ T_1 &= 01:00:00 - 2.83991567h = 22:09:36, \text{ June } 5 \\ Q_1 &= -1.8269125 + 2.53652518 = 0.70961268 = 41^\circ \end{aligned}$$

$$\begin{aligned} \therefore \psi_4 &= \arcsin(0.56881757) = 0.60506747 \\ \tau_4 &= \frac{(0.00458491 + 0.00014012)\cos(0.60506747) - 0.00274876\cos(1.615708 - (-2.88547645))}{0.0011655} \\ &= 3.82875144 \\ T_4 &= 01:00:00 + 3.82875143h = 04:49:44 \\ Q_4 &= -1.8269125 + 0.60506747 = -1.22184503 = -70^\circ \end{aligned}$$

Now for interior contacts:

$$\begin{aligned}
\sin(\psi) &= \frac{0.00274876\sin(-0.04491168-(-1.8269125))}{0.00458491-0.00014012} \\
&= 0.60468101 \\
\therefore \psi_2 &= \pi - \arcsin(0.60468101) = 2.49222735 \\
\tau_2 &= \frac{(0.00458491-0.00014012)\cos(2.49222735)-0.00274876\cos(1.615708-(-2.88547645))}{0.0011655} \\
&= -2.54301832 \\
T_2 &= 01:00:00 - 2.54301832h = 22:27:25, \text{ June 5} \\
Q_2 &= -1.8269125 + 2.49222735 = 0.66531485 = 38^\circ
\end{aligned}$$

$$\begin{aligned}
\therefore \psi_3 &= \arcsin(0.60468101) = 0.6493653 \\
\tau_3 &= \frac{(0.00458491-0.00014012)\cos(0.6493653)-0.00274876\cos(1.615708-(-2.88547645))}{0.0011655} \\
&= 3.53185408 \\
T_3 &= 01:00:00 + 3.53185408h = 04:31:54 \\
Q_3 &= -1.8269125 + 0.6493653 = -1.1775472 = -67^\circ
\end{aligned}$$

Greatest transit occurred at (by equations 9.85 and 9.86):

$$T_{G.T.} = 01:00:00 - \frac{0.00274876\cos(1.615708-(-2.88547645))}{0.0011655} = 01:29:40$$

And the minimum separation was (by equation 9.87):

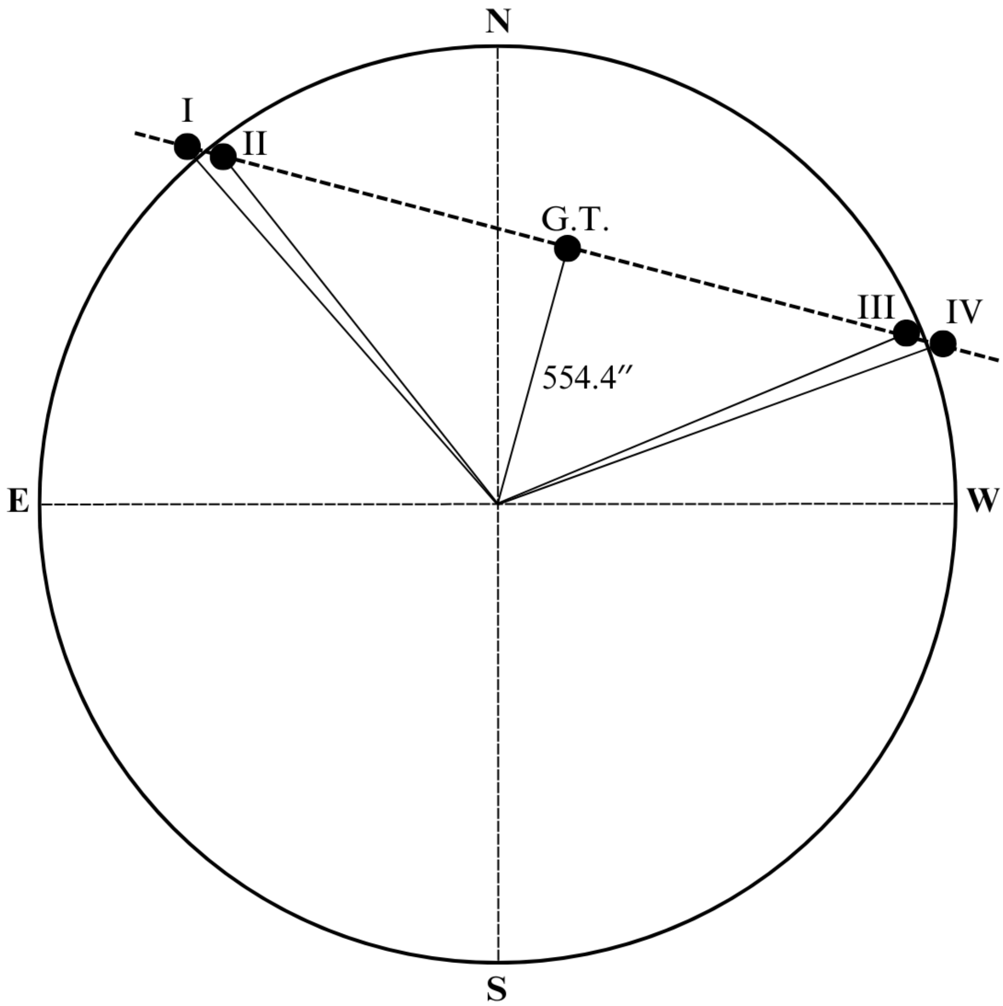
$$\Delta = |0.00274876\sin(1.615708-(-2.88547645))| = 0.00268768 \text{ rad} = 554.4''$$

And the position angle by was (by equation 9.88):

$$Q = -1.8269125 + \pi/2 = -0.25611617 = -15^\circ$$

Our calculations can be summarized in this diagram:

Transit of Venus across the Sun on June 5/6, 2012
Transit Duration $= 6h\ 40m$ Min. Sep. $= 554.4''$



Contact	Time	Pos. Angle
I	22:09:36	41°
II	22:27:25	38°
G.T.	01:29:40	345°
III	04:31:54	293°
IV	04:49:44	290°